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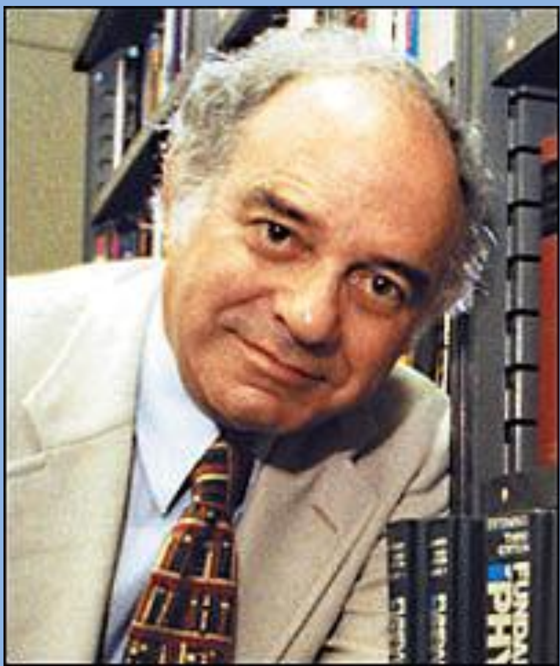

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Ignazio Licata **Ammar Sakaji**

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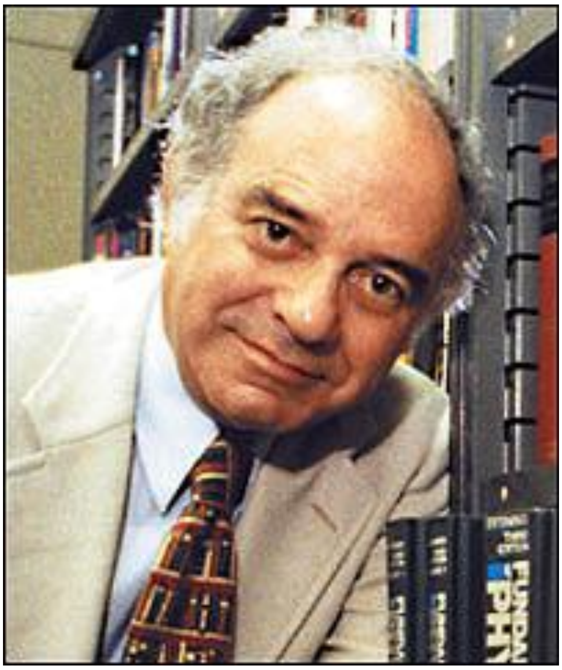

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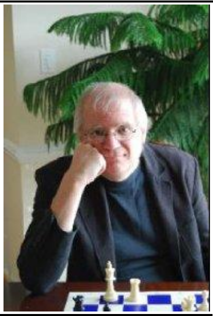

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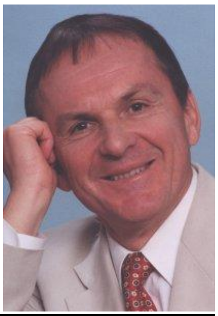

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Double Field Theory Symmetries of the String and the Weak Scale

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Abstract: This work focuses on understanding the symmetries behind double field theory and how to reduce the theory to the usual one with Lorentz symmetry and T-duality. Although this work has mainly the structure of a review, new ideas are explored and old results are taken from a new point of view. Considering the physical effects of dual dimensions, a connection with the weak scale is performed through compactification.

Keywords: Double Field Theory; Lorentz Symmetry; T-duality; Branes; Symmetry
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1. Introduction

The T-duality group $O(d, d; \mathbb{Z})$ is well known to emerge when closed string theory is put on a torus background. In order to understand the recent development of a formulation that makes this symmetry manifest at the level of a spacetime action, this work starts by introducing some basic notions based mainly on [1]. For nice reviews, consult [1], [2], [3] and reference therein.

The total number of dimensions is denoted by D. When compactification is considered, d will denote the number of compact dimensions and n the remaining dimensions. Of course, the main interest is in $D = 10$, $d = 6$ and $n = 3+1$, but since most computations can be performed without explicitly specifying the dimensions, the results are kept as general as possible. When necesary, the string lenght $l_s = 2\pi\sqrt{\alpha'}$ will be used. Its inverse $M_s = l_s^{-1}$ is the string scale, it represents the energy scale at which is believed string theory effects would first make themselves felt.

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1.1 Toroidal Compactification of Closed String

T-duality relates winding modes in a given compact space with momentum modes in another (dual) compact space for a closed string. The action governing the worldsheet dynamics of closed string in a background reads:

$$S = -\frac{1}{4\pi\alpha'}\int_{\Sigma}\left(\eta^{\alpha\beta}\partial_\alpha X^i\partial_\beta X^j G_{ij} + \epsilon^{\alpha\beta}\partial_\alpha X^i \partial_\beta X^j B_{ij}\right) d\sigma d\tau \ . \tag{1}$$

The integration is over the cylinder Σ parameterized by $(\sigma,\tau)\in[0,2\pi]\times[\tau_0,\tau_f]$. The X^a are periodic coordinates for the compact dimensions, $X^i=(X^a,X^\mu)$, $\eta^{\alpha\beta}=\mathrm{diag}(-1,1)$, $\epsilon^{01}=-1$ and $\partial_\alpha=(\partial_\tau\,,\partial_\sigma)$. The coupling to B_{ij} is the natural generalization to worldsheets of the coupling of a vector gauge field A_j to the worldline of a particle. That means that strings carry Kalb-Ramond charge.

Both G and B are constant and can be combined into the field E defined by $E_{ij} = G_{ij} + B_{ij}$. The expansion of the string coordinate in terms of momenta, windings, and oscillators is:

$$X^i(\tau,\sigma) = x_0^i + \alpha' G^{ij}(p_j - B_{jk}w^k)\tau + \alpha' w^i \sigma + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{e^{-in\tau}}{n}[\bar{\alpha}_n^i e^{-in\sigma} + \alpha_n^i e^{in\sigma}]. \tag{2}$$

The zero modes α_0 and $\bar{\alpha}_0$ are given by:

$$\begin{aligned} \alpha_0^i &= \sqrt{\frac{\alpha'}{2}}G^{ij}(p_j - E_{jk}w^k)\,, \\ \bar{\alpha}_0^i &= \sqrt{\frac{\alpha'}{2}}G^{ij}(p_j + E_{kj}w^k)\,. \end{aligned} \tag{3}$$

Also there is a generalized metric constructed out of G and B:

$$\begin{aligned} \mathcal{H} &= \begin{pmatrix} G - BG^{-1}B & BG^{-1} \\ -G^{-1}B & G^{-1} \end{pmatrix} = \begin{pmatrix} 1 & B \\ 0 & 1 \end{pmatrix}\begin{pmatrix} G & 0 \\ 0 & G^{-1} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -B & 1 \end{pmatrix}, \\ \mathcal{H}^{-1} &= \begin{pmatrix} G^{-1} & -G^{-1}B \\ BG^{-1} & G - BG^{-1}B \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix}\begin{pmatrix} G^{-1} & 0 \\ 0 & G \end{pmatrix}\begin{pmatrix} 1 & -B \\ 0 & 1 \end{pmatrix}. \end{aligned} \tag{4}$$

This $2D\times 2D$ symmetric matrix can be written with indices $\mathcal{H}^{MN}$ with $M,N = 1,\ldots,2D$ and its inverse as $\mathcal{H}_{MN}$.

1.2 Extending Spacetime

Invariance of the physics under background transformations is a major aspect to understand T-duality. N and $\bar{N}$ are the number operators counting the right and the left moving excitations in any state. In term of the Virasoro operators with zero mode

number $L_0 = \frac{1}{2}\alpha_0^i G_{ij}\alpha_0^j + N$, $\bar{L}_0 = \frac{1}{2}\bar{\alpha}_0^i G_{ij}\bar{\alpha}_0^j + \bar{N}$, the level matching condition on the spectrum gives $N - \bar{N} = p_i w^i$, or equivalently:

$$N - \bar{N} = \frac{1}{2}P^t \eta P \ , \tag{5}$$

where P and η are:

$$P = \begin{pmatrix} w^i \\ \\ p_i \end{pmatrix} \ , \qquad \eta = \begin{pmatrix} 0 & I \\ \\ I & 0 \end{pmatrix} \ . \tag{6}$$

Transformations leaving η invariant constitute the $O(D, D)$ group.

Massless fields with $N = \bar{N} = 0$ have the form:

$$\textstyle\sum_{p,w} \ e_{ij}(p, w)\alpha_{-1}^i \bar{\alpha}_{-1}^j c_1 \bar{c}_1 |p, w\rangle, \tag{7}$$

$$\textstyle\sum_{p,w} \ d(p, w)(c_1 c_{-1} - \bar{c}_1 \bar{c}_{-1})|p, w\rangle \ , \tag{8}$$

with momentum space wavefunctions $e_{ij}(p, w)$ and $d(p, w)$. Gravitational fluctuations plus Kalb-Ramond fluctuations are encoded in e_{ij} as a background fluctuation, and the usual dilaton field ϕ is related nonlinearly with d by $e^{-2d} = e^{-2\phi}\sqrt{-g}$. Fourier transforming to position space then gives component fields that depend on both the spacetime coordinates and on new periodic coordinates conjugate to winding number. Written with $p_i = \frac{1}{i}\frac{\partial}{\partial x^i}$ and $w^i = \frac{1}{i}\frac{\partial}{\partial \tilde{x}_i}$ the zero modes are given by:

$$\alpha_{0i} = -\sqrt{\tfrac{\alpha'}{2}}\left(\tfrac{\partial}{\partial x^i} - E_{ik}\tfrac{\partial}{\partial \tilde{x}_k}\right) \equiv -\tfrac{i}{\sqrt{2}}D_i \ , \tag{9}$$

$$\bar{\alpha}_{0i} = -\sqrt{\tfrac{\alpha'}{2}}\left(\tfrac{\partial}{\partial x^i} + E_{ki}\tfrac{\partial}{\partial \tilde{x}_k}\right) \equiv -\tfrac{i}{\sqrt{2}}\bar{D}_i \ , \tag{10}$$

where $D_i = \partial_i - E_{ik}\tilde{\partial}^k$, $D^i \equiv G^{ij}D_j$ and also $\bar{D}_i = \partial_i + E_{ki}\tilde{\partial}^k$, $\bar{D}^i \equiv G^{ij}\bar{D}_j$. One can also show that $\frac{1}{2}(D^i D_i - \bar{D}^i \bar{D}_i) = -2\partial_i\tilde{\partial}^i$ and the constraint (5) can be written as:

$$N - \bar{N} = -\partial_i\tilde{\partial}^i \equiv -\partial \cdot \tilde{\partial} \ , \tag{11}$$

so it is useful to define also:

$$\square \equiv D^i D_i = \bar{D}^i \bar{D}_i \ . \tag{12}$$

On account of this, fields $e_{ij}(x, \tilde{x})$ and $d(x, \tilde{x})$ must satisfy the constraint:

$$\partial \cdot \tilde{\partial} \ e_{ij}(x, \tilde{x}) = \partial \cdot \tilde{\partial} \ d(x, \tilde{x}) = 0 \ . \tag{13}$$

All this leads to motivation to extend spacetime and use coordinates $X^M = (\tilde{x}_i, x^i)$ in order to rearrange the degrees of freedom in the T-duality invariant formulation of Double Field Theory (DFT). Everything must be in $O(D, D)$ representation. The coordinates here can either parameterize compact or non-compact directions indistinctively. Even if non-compact, one can still formulate a full $O(D, D)$ covariant theory. The duals to the non-compact directions are just ineffective, and one can assume that nothing

depends on them. For the theory to be consistent, $\partial^M \partial_N$ is constrained to vanish for all products of fields and gauge parameters:

$$\eta^{MN} \partial_M \phi'(X) \partial_N \phi''(X) = 0 \ . \tag{14}$$

This *strong constraint* goes beyond the level-matching constraint of closed string theory and implies that the fields depend only on half of the doubled coordinates.

1.3 Invariant action

First, an action with a background field $E_{i\bar{j}}$ and small fluctuations $e_{i\bar{j}}(x, \tilde{x})$ were considered in [4]:

$$S = \int dx\, d\tilde{x} \mathcal{L}(e_{ij}, D^i e_{ij}, \bar{D}^j e_{ij}, D^i \bar{D}^j e_{ij}, \Box e_{ij}, D_i D^k e_{kj}, \bar{D}_j \bar{D}^k e_{ik}, d, D^i \bar{D}^j d, \Box d) \ . \tag{15}$$

The action was obtained to cubic order in the fields. This should be thought of as a background which contains a gravitational background as well as a Kalb-Ramond background and a dilaton $d(x, \tilde{x})$. However, one is looking for a manifest background independent version [5] which does not rely on an explicit distinction between the background field E_{ij} and its fluctuation. The field $\mathcal{E}_{ij}(X) = E_{ij} + e_{ij}(x, \tilde{x}) + O(e^2)$ is introduced to stress the point of background independence, which at the linearized level is the sum of E_{ij} and e_{ij}.

After analizing how $\mathcal{E}$ and $\mathcal{D}$ behave under T-duality the full background independent $O(D, D)$ action for the fields $\mathcal{E}$ and d is given by:

$$\begin{aligned} S_{\mathcal{E},d} = \int dx\, d\tilde{x}\, e^{-2d} \Big[-\frac{1}{4}\, g^{ik} g^{j\ell} \mathcal{D}^p \mathcal{E}_{k\ell} \mathcal{D}_p \mathcal{E}_{ij} + \frac{1}{4}\, g^{k\ell} \left(\mathcal{D}^j \mathcal{E}_{ik}\, \mathcal{D}^i \mathcal{E}_{j\ell} + \bar{\mathcal{D}}^j \mathcal{E}_{ki}\, \bar{\mathcal{D}}^i \mathcal{E}_{\ell j} \right) \\ + \left(\mathcal{D}^i d\, \bar{\mathcal{D}}^j \mathcal{E}_{ij} + \bar{\mathcal{D}}^i d\, \mathcal{D}^j \mathcal{E}_{ji} \right) + 4\, \mathcal{D}^i d\, \mathcal{D}_i d \Big]. \end{aligned} \tag{16}$$

Each term is independently $O(D, D)$ invariant. It is desirable to express everything in terms of the generalized metric only. For example, the dilaton term $4\mathcal{D}^i d \mathcal{D}_i d$. A contraction with η is not reasonable, since then the strong constraint will kill it. The only other possibility is to contract the indices with $\mathcal{H}$, yielding a term $4\mathcal{H}^{MN} \partial_M d \partial_N d$. This does not only work for the dilaton term, but also all other terms in this action can be rewritten in this way:

$$\begin{aligned} S_{\mathcal{H},d} = \int dx\, d\tilde{x}\, e^{-2d} \Big(\frac{1}{8}\, \mathcal{H}^{MN} \partial_M \mathcal{H}^{KL} \partial_N \mathcal{H}_{KL} - \frac{1}{2}\, \mathcal{H}^{MN} \partial_N \mathcal{H}^{KL} \partial_L \mathcal{H}_{MK} \\ -2\, \partial_M d\, \partial_N \mathcal{H}^{MN} + 4\, \mathcal{H}^{MN} \partial_M d\, \partial_N d \Big) \ . \end{aligned} \tag{17}$$

An $O(D, D)$ covariant action can be cast in Einstein-Hilbert form for a scalar function of the generalised metric and doubled dilaton which is denoted $\mathcal{R}$. The notation makes clear that in some sense the scalar is playing the role that the Ricci scalar does in ordinary gravity. The derivation of such an action is lengthy and here is the result:

$$S = k \int dx\, d\tilde{x}\, e^{-2d}\, \mathcal{R}(\mathcal{H}, d) \ . \tag{18}$$

In order to obtain dimensions of action, k must have L^{2-2D} dimension. The dilaton equation of motion is the vanishing of $\mathcal{R}$ and the equation of motion for $\mathcal{H}$ is:

$$R_{MN} = 0 \ , \tag{19}$$

where R_{MN} is a generalized Ricci tensor. More interesting is that conventional low-energy action (the universal massless bosonic sector of supergravity) can be obtained from (18) setting $\tilde{\partial} = 0$:

$$S|_{\tilde{\partial}=0} = k\tilde{V} \int d^D x \sqrt{g} e^{-2\phi} \left[R + 4(\partial\phi)^2 - \frac{1}{12} H^{ijk} H_{ijk} \right] \ . \tag{20}$$

Important to note the volume $\tilde{V}$ of the dual space which arise in the $\tilde{x}$ integration. In this way DFT can be viewed as a T-duality covariant formulation of the NS-NS sector of supergravity.

Extension to the RR sector

The RR sector for type IIA consists of a 1-form and a 3-form. Section 5 in this work will focus on type IIB whose bosonic massless fields consist of a scalar C_0, a 2-form C_2 and a 4-form C_4 with field strengths $F_1 = dC_0$, $F_3 = dC_2$ and $F_5 = dC_4$. A democratic formulation [6] that simultaneously uses dual forms, such that type IIA contains all odd forms, and type IIB contains all even forms, both being supplemented by duality relations is the clue to how RR fields may be included into the DFT formulation [7]. Using a general spinor state and a suitable representation $\mathbb{S}$ of $\mathcal{H}$ in the $Spin^-(D, D)$ group, the action reads:

$$S = k \int dx d\tilde{x} \left(e^{-2d} \mathcal{R}(\mathcal{H}, d) + \frac{1}{4} (\partial\!\!\!/ \chi)^\dagger \, \mathbb{S} \, \partial\!\!\!/ \chi \right) , \tag{21}$$

supplemented by the constraint $\partial\!\!\!/ \chi = -C^{-1} \mathbb{S} \, \partial\!\!\!/ \chi$, where C is the charge conjugation matrix. Upon setting $\tilde{\partial} = 0$ one does indeed find that this gives the correct kinetic terms for the RR fields, all even or all odd forms, depending on the chirality of χ. The democratic formulation of type II theories, whose field equations are equivalent to the conventional field equations of type IIA for odd forms and of type IIB for even forms, can be obtained in this way.

2. Double sigma models

Consider the closed string action without the Kalb-Ramond term, in a flat spacetime:

$$S = -\frac{1}{4\pi\alpha'} \int_\Sigma \eta^{\alpha\beta} \partial_\alpha X^i \partial_\beta X^j G_{ij} d\sigma d\tau \ . \tag{22}$$

The integrand must be single-valued on Σ parameterized by $(\sigma, \tau) \in [0, 2\pi] \times [\tau_0, \tau_f]$, this implies that $\partial_\tau X^i$ and $\partial_\sigma X^i$ have to be periodic with respect to σ with period 2π.

However, this does not mean that $X^i(\sigma,\tau)$ must be a periodic function. Instead, it means that X^i must be a quasi-periodic function which satisfies:

$$X^i(\sigma+2\pi,\tau) = X^i(\sigma,\tau) + \xi^i. \tag{23}$$

Here ξ^i is the quasi-period of X^i. If ξ^i is not zero, there is no a geometrical interpretation of a closed string propagating in a flat spacetime. Of course, if X^i were compact and spacelike, then ξ^i would be interpreted as winding, and it is not in general zero. Substracting $X^i(\sigma,\tau)$ in (23), and for $\sigma = 0$, the quasi-period can be expresed performing an integration along the string:

$$\int_0^{2\pi} \partial_\sigma X^i d\sigma = \xi^i \ . \tag{24}$$

Also, it is known that the momentum density $\mathcal{P}^i$ and $\partial_\tau X^i$ are related by:

$$\frac{1}{2\pi\alpha'}\partial_\tau X^i = \mathcal{P}^i \ . \tag{25}$$

So, then the momentum of the string is:

$$\int_0^{2\pi} \mathcal{P}^i = p^i = \frac{1}{2\pi\alpha'}\int_0^{2\pi} \partial_\tau X^i d\sigma \ . \tag{26}$$

This suggests that when the quasiperiod is interpreted as winding the expression $\frac{1}{2\pi\alpha'}\partial_\sigma X^i$ could be interpreted as a winding density. Therefore, winding should arise performing the integration along the string:

$$\frac{1}{2\pi\alpha'}\int_0^{2\pi} \partial_\sigma X^i d\sigma = w^i \ . \tag{27}$$

So (23) can be rewritten in a more familiar way:

$$X^i(\sigma+2\pi,\tau) = X^i(\sigma,\tau) + 2\pi\alpha' w^i. \tag{28}$$

Equation (26) also suggests that one could look for a function which its sigma-derivative can be replaced by $\partial_\tau X^i$ making momentum the quasi-period of this function.

Winding can be seen as the conjugate momentum associated with a dual coordinate. What is momentum for $\tilde{X}$ is winding for X, and what is winding for $\tilde{X}$ is momentum for X. So, then the winding density $\frac{1}{2\pi\alpha'}\partial_\sigma \tilde{X}^i$ associated to $\tilde{X}^i$ leads to momentum:

$$p^i = \frac{1}{2\pi\alpha'}\int_0^{2\pi} \partial_\sigma \tilde{X}^i d\sigma \ . \tag{29}$$

This implies that $\tilde{X}^i$ satisfies: $\tilde{X}^i(2\pi,\tau) = \tilde{X}^i(0,\tau) + 2\pi\alpha' p^i$. The string can be parameterized by $[\sigma, \sigma+2\pi]$, not just for $\sigma = 0$, in X and also in $\tilde{X}$, so X^i satisfies:

$$\tilde{X}^i(\sigma+2\pi,\tau) = \tilde{X}^i(\sigma,\tau) + 2\pi\alpha' p^i \ . \tag{30}$$

All this is consistent with all known about basic T-duality.

Now, as suggested by the preceding discussion, a dual space coordinate $\tilde{X}$ will be introduced such that:

$$\partial_\sigma \tilde{X}^i = 2\pi\alpha' \mathcal{P}^i. \tag{31}$$

Like X^i, the coordinate $\tilde{X}^i$ is not periodic, its quasi-period $\tilde{X}^i(2\pi,\tau) - \tilde{X}^i(0,\tau)$ is propotional to the string momentum. Now, the first order action will be used as a guide in order to build a double sigma model:

$$S = \int_\Sigma [\mathcal{P}_j \partial_\tau X^j - \frac{1}{2}(2\pi\alpha' \mathcal{P}_j \mathcal{P}^j + \frac{1}{2\pi\alpha'}\partial_\sigma X_j \partial_\sigma X^j)] d\tau d\sigma \ . \tag{32}$$

Note that the target space of the sigma model is being doubled.

Using $\mathcal{P}^i = \frac{1}{2\pi\alpha'}\partial_\sigma \tilde{X}^i$ the action becomes:

$$S = \frac{1}{2\pi\alpha'}\int_\Sigma (\partial_\sigma \tilde{X}_j \partial_\tau X^j + \frac{1}{2}(\partial_\sigma \tilde{X}_j \partial_\sigma \tilde{X}^j + \partial_\sigma X^j \partial_\sigma X_j)) d\tau d\sigma \ . \tag{33}$$

In the action that is being constructed the main point is that both X and $\tilde{X}$ are taken to be quasi-periodic and are treated on an equal footing. The first term in (33) is not symmetric under $X \to \tilde{X}$, $\tilde{X} \to X$, therefore the following replacement is proposed:

$$\partial_\sigma \tilde{X}_j \partial_\tau X^j \to \frac{1}{2}(\partial_\sigma \tilde{X}_j \partial_\tau X^j + \partial_\sigma X^j \partial_\tau \tilde{X}_j) \ . \tag{34}$$

This replacement gives:

$$S = \frac{1}{4\pi\alpha'}\int_\Sigma (\partial_\sigma \tilde{X}_j \partial_\tau X^j + \partial_\sigma X^j \partial_\tau \tilde{X}_j + \partial_\sigma \tilde{X}_j \partial_\sigma \tilde{X}^j + \partial_\sigma X^j \partial_\sigma X_j) d\tau d\sigma \ . \tag{35}$$

The usual formulation is recovered if one insists that X is single-valued, and the usual T-duality is recovered if one insists that quasi-periods of X appear only along space-like directions and have only discrete values.

Using the matrices:

$$H_{MN} = \begin{pmatrix} G^{-1} & 0 \\ 0 & G \end{pmatrix} \quad , \qquad \eta = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \ , \tag{36}$$

the action can be written in a more familiar way to the DFT formalism:

$$S = \frac{1}{4\pi\alpha'}\int_\Sigma (\eta_{MN}\partial_\tau X^M \partial_\sigma X^M - H_{MN}\partial_\sigma X^M \partial_\sigma X^M) d\tau d\sigma \ , \tag{37}$$

where the coordinate X^M is given by:

$$X^M(\tau,\sigma) = \begin{pmatrix} \tilde{X}_i(\tau,\sigma) \\ X^i(\tau,\sigma) \end{pmatrix} \ . \tag{38}$$

Assuming $O(D,D)$ symmetry, a transformation can be done to obtain B fields. That means one can write the action using the generalized metric instead of H_{MN}:

$$S = \frac{1}{4\pi\alpha'}\int_\Sigma (\eta_{MN}\partial_\tau X^M \partial_\sigma X^M - \mathcal{H}_{MN}\partial_\sigma X^M \partial_\sigma X^M) d\tau d\sigma \ . \tag{39}$$

In this way, the action takes a form previously considered by Tseytlin[8]. Now, generalizations are proposed. First, $\mathcal{H}$ is allowed to depend on both X and $\tilde{X}$ irrespective of whether the directions are flat or non-compact. Second, non-trivial quasi-periods both along X and $\tilde{X}$ are allowed even when they label non-compact or timelike directions. The conclusions reached from some of these basic assumptions may reformulate string theory in a larger context.

If $\mathcal{H}$ depends only on X along non-compact directions, quasi-periods are only along $\tilde{X}$, except when there are compact flat directions, then this action is equivalent to the usual formulation on an arbitrary curved non-compact manifold or on a flat torus bundle over a non-compact curved space. To find solutions in a general way is an interesting question. Expansions of X and $\tilde{X}$ for the model in section 1.1 read:

$$X^i = x_0^i + \alpha' G^{ij}(p_j - B_{jk}w^k)\tau + \alpha' w^i \sigma + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{e^{-in\tau}}{n}[\bar{\alpha}_n^i e^{-in\sigma} + \alpha_n^i e^{in\sigma}] \ , \qquad (40)$$

$$\begin{aligned}\tilde{X}_i = \tilde{x}_{0i} &+ \alpha'[(G - BG^{-1}B)_{ij}w^j + (BG^{-1})_i^j p_j]\tau + \alpha' p_i \sigma \\ &+ i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{e^{-in\tau}}{n}[E_{ij}\bar{\alpha}_n^j e^{-in\sigma} - E_{ji}\alpha_n^j e^{in\sigma}] \quad .\end{aligned} \qquad (41)$$

These can be put in what should be the expansion of the solution of (39) by construction:

$$\begin{aligned}X^M = &\begin{pmatrix} x_0^i \\ \tilde{x}_{0i} \end{pmatrix}^M + \alpha' \mathcal{H}^{MN} P_N \tau + \alpha' P^M \sigma \\ &+ i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{e^{-in\tau}}{n} \left[\begin{pmatrix} \bar{\alpha}_n^i \\ E_{ij}\bar{\alpha}_n^j \end{pmatrix}^M e^{-in\sigma} + \begin{pmatrix} \alpha_n^i \\ -E_{ji}\alpha_n^j \end{pmatrix}^M e^{in\sigma} \right] \ .\end{aligned} \qquad (42)$$

An important aspect is to find the background conditions that preserve worldsheet Weyl invariance. Expanding X^M as a classical part plus a quantum fluctuation, $X^M = X^M_{cl} + \delta X^M$, and replacing in the action one expect to obtain, at second order in fluctuation, the one loop beta function from S^{1-loop}:

$$S(X_{cl} + \delta X) = S_{cl} + \int (\beta_{MN}(\mathcal{H})\partial_1 X^M \partial_1 X^N) d\tau d\sigma \ . \qquad (43)$$

It is through this function that the connection to double field theory is performed in [9]. The result for the calculation of S^{1-loop} is:

$$S^{1-loop} = \int (R_{MN}\partial_1 X^M \partial_1 X^N) d\tau d\sigma \ . \qquad (44)$$

Given the result for the one-loop effective action we can find the equation of motion in the same manner as for the ordinary string:

$$\beta_{MN}[\mathcal{H}] = R_{MN} = 0 \ . \qquad (45)$$

The vanishing of the one loop beta function is equivalent to the generalised metric equation of motion of doubled field theory given by equation (19). Double field theory is the effective action for the new sigma model.

In the same way as ordinary effective ations are provided by a constant $k = \frac{2\pi}{l_s^{D-2}}$, in the doubled case (18) the constant should yield the action:

$$S = \frac{2\pi}{l_s^{2D-2}} \int dx\, d\tilde{x}\, e^{-2d}\, \mathcal{R}(\mathcal{H}, d) \ . \tag{46}$$

New kinds of generalizations are possible if the condition on η to be a constant is relaxed, namely it is allowed it to be arbitrary curved. It is not clear, of course, that these sigma models are consistent and consistency conditions coming from the quantum dynamics must be looked for in future research.

3. Double Field Theory Symmetries

As pointed in (4) the components of the generalized metric form a matrix that is the product of three matrices. Each one of these is an $O(D, D)$ matrix. Descomposing the metric g as $g_{ij} = (e)_i^a \eta_{ab} (e^t)_j^b$, where e appears like a vielbein, ($\eta_{ab} = \mathrm{diag}(-1, 1, 1, 1)$ for $D = 4$ for example) one can do the splitting:

$$\begin{pmatrix} g & 0 \\ 0 & g^{-1} \end{pmatrix} = \begin{pmatrix} e & 0 \\ 0 & (e^t)^{-1} \end{pmatrix} \hat{\eta} \begin{pmatrix} e^t & 0 \\ 0 & e^{-1} \end{pmatrix} \ , \tag{47}$$

where:

$$\hat{\eta} = \begin{pmatrix} \eta_{ab} & 0 \\ 0 & \eta^{ab} \end{pmatrix} \ . \tag{48}$$

Defining:

$$\mathrm{V} = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e & 0 \\ 0 & (e^t)^{-1} \end{pmatrix} \ , \tag{49}$$

the matrix $\mathcal{H}$ can be written as:

$$\mathcal{H} = \mathrm{V} \hat{\eta} \mathrm{V}^t \ . \tag{50}$$

So $\mathcal{H}$ is parameterized by V, which of course takes values on $O(D, D)$, but V is ambiguous since it is always possible to replace V by Vh where:

$$h \hat{\eta} h^t = \hat{\eta} \ . \tag{51}$$

These h are elements of the subgoup $O(D-1, 1) \times O(D-1, 1)$, so $\mathcal{H}$ is taking values on the coset manifold:

$$\mathcal{M} = O(D, D)/O(D-1, 1) \times O(D-1, 1). \tag{52}$$

Distinct backgrounds are characterised by the points of $\mathcal{M}$.

The $O(D,D)$ symmetry means that over the double space there is a line element:

$$ds^2 = dX^M \eta_{MN} dX^N = 2d\tilde{x}_j dx^j \ , \tag{53}$$

with this metric a product can be built. Of course, this symmetry is too general and one wonders how to obtain compatible symmetries with physical solutions. First, the invariance of physics constraint is going to be checked for the solution found in section 2.. The physical state conditions $L_0 - 1 = 0$ and $\bar{L}_0 - 1 = 0$ can be written as:

$$L_0 - 1 = \tfrac{1}{2}\alpha_0^i G_{ij} \alpha_0^j + N - 1 = 0 \ , \tag{54}$$

$$\bar{L}_0 - 1 = \tfrac{1}{2}\bar{\alpha}_0^i G_{ij} \bar{\alpha}_0^j + \bar{N} - 1 = 0 \ , \tag{55}$$

As usual the zero mode oscillators will be identified with left and right momenta:

$$\alpha_0^i = \sqrt{\tfrac{\alpha'}{2}} G^{ij}(p_j - E_{jk} w^k) = \sqrt{\tfrac{\alpha'}{2}} P_R \ , \tag{56}$$

$$\bar{\alpha}_0^i = \sqrt{\tfrac{\alpha'}{2}} G^{ij}(p_j + E_{kj} w^k) = \sqrt{\tfrac{\alpha'}{2}} P_L \ , \tag{57}$$

the novel aspect here is that the descomposition in left and right momenta is allowed to all dimensions. For the right momentum:

$$-P_R^j G_{ij} {P_R}^j = \frac{4}{\alpha'}(N-1) \ , \tag{58}$$

and for the left excitations:

$$-P_L^j G_{ij} {P_L}^j = \frac{4}{\alpha'}(\bar{N}-1) \ . \tag{59}$$

After some calculation, one can realize that:

$$P_R^j P_{Rj} + P_L^j P_{Lj} = 2P^M \mathcal{H}_{MN} P^N \ , \tag{60}$$

$$-P_R^j P_{Rj} + P_L^j P_{Lj} = 2P^M P_M \ . \tag{61}$$

The constraints for the sigma model considered in section 2. can be obtained in an $O(D,D)$ invariant way:

$$-P^M \mathcal{H}_{MN} P^N = \frac{2}{\alpha'}(N + \bar{N} - 2) \ , \tag{62}$$

$$\frac{\alpha'}{2} P^M P_M = N - \bar{N} \ . \tag{63}$$

There is another symmetry, $O(d,d;\mathbb{Z})$, that will arise when periodicity conditions are imposed to certain coordinates. Finally Lorentz symmetry, a more common one to real world, is also a subsymmetry of $O(D,D)$.

4. Breaking

When formulate in non-compact dimensions the double sigma model has an $O(D,D)$ symmetry, so there is a line element ruling the x and $\tilde{x}$ coordinates. There is another invariant line element which can be constructed with $\mathcal{H}$:

$$ds^2 = dX^M \mathcal{H}_{MN} dX^N \ . \tag{64}$$

Is from this invariant that Lorentz symmetry will emerge.

Going to a frame where $B = 0$, this can be done if a suitable $O(D,D)$ transformation is performed[2], the line element becomes:

$$ds^2 = dx^i g_{ij} dx^j + d\tilde{x}_i g^{ij} d\tilde{x}_j \ . \tag{65}$$

Restricting the fields to be independent of $\tilde{x}_i$, with the strong constraint for example, the first line element vanishes:

$$ds^2 = dX^M \eta_{MN} dX^N = 0 \ . \tag{66}$$

This does not say anything about the symmetry wanted. But the other line element is reduced to:

$$ds^2 = dx^i g_{ij} dx^j \ . \tag{67}$$

So, Lorentz symmetry is not emerging from the first line element but for the other line element. In this way the breaking of $O(D,D)$ into $O(D-1,1)$ has been obtained. However, it is important to stress that the fundamental line element can be written in the usual way only in certain $O(D,D)$ frames.

Now, going back to the theory with full $O(D,D)$ symmetry, the case with d compact dimensions will be considered. First, is valuable to understand that $O(n,n) \times O(d,d)$ is a subgroup of $O(D,D)$, it makes trasformations separately on the line element in (53) expressed as:

$$ds^2 = dX^M \eta_{MN} dX^N = 2d\tilde{x}_\mu dx^\mu + 2d\tilde{x}_a dx^a \ , \tag{68}$$

where $O(n,n)$ acts on $2d\tilde{x}_\mu dx^\mu$ and $O(d,d)$ acts on $2d\tilde{x}_a dx^a$. If periodicity conditions over x are imposed the symmetries will be restricted. This can be seen performing a transformation on the $O(d,d)$ vector of windings and momenta:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} w \\ p \end{pmatrix} = \begin{pmatrix} w' \\ p' \end{pmatrix} \ . \tag{69}$$

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[2] As $\mathcal{H} = \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix} \begin{pmatrix} g^{-1} & 0 \\ 0 & g \end{pmatrix} \begin{pmatrix} 1 & -b \\ 0 & 1 \end{pmatrix}$, it is easy to see that $\mathcal{H}' = R^t \mathcal{H} R$, with transformation $R = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$, is a matrix of the form $\mathcal{H}' = \begin{pmatrix} g^{-1} & 0 \\ 0 & g \end{pmatrix}$. Therefore, always is possible to find a suitable transformation to remove the Kalb-Ramond field.

The periodicity condition on x demand p to take discrete values. The relation:

$$p' = cw + dp \ , \tag{70}$$

must be satisfied, then c, w and d should take discrete values in order to ensure discrete values to p'. Also $w' = aw + bp$, then a, b and w' shall take discrete values. If w' is discrete, then $\tilde{x}$ is periodic. This induces compactification on the other coordinates, as seen in (70), they are linked. Therefore, compactifying d of the dimensions on a torus breaks this to $O(n,n) \times O(d,d;\mathbb{Z})$.

So now the breaking $O(n,n) \to O(n-1,n)$ is completely analogous to the breaking $O(D,D) \to O(D-1,1)$ seen above. Therefore, these breakings have been shown as claimed in [5].

Now, consider equations (62) and (63). The breaking of these relations leads to the usual relations:

$$-p^\mu p_\mu = \begin{pmatrix} w^a & p_b \end{pmatrix} \begin{pmatrix} G^{ab} & -G^{ac}B_{cb} \\ B_{ac}G^{cb} & G_{ab} - B_{ac}G^{cd}B_{db} \end{pmatrix} \begin{pmatrix} w^b \\ p_b \end{pmatrix} + \frac{2}{\alpha'}(N + \bar{N} - 2), \tag{71}$$

$$\alpha' \tilde{p}^a p_a = N - \bar{N} \ , \tag{72}$$

which was the expected result.

5. Effects on the mass scales derived from compactification

The possibility of explain the weakness of gravity has been explored extensively [10] motivated by the string theoretic expression:

$$M_P^2 = \frac{1}{g_s^2} M_s^2 V_6 \ , \tag{73}$$

relating the string scale, the Planck mass M_P, the six dimensional internal volume V_6 and the string coupling $g_s = e^{\phi_0}$. Keeping M_s fixed the Planck scale can be accounted for in two distinct ways:

(1)The large size of the volume of extra dimensions V_6, while keeping g_s of order unity.

(2)The other possibility arises, keeping the string scale near the TeV scale, when size of the extra dimensions are at a TeV and attributing the enormity of the Planck mass to a tiny $g_s \sim 10^{-16}$. The hierarchy problem is equivalent to understanding the smallness of g_s, this can be achieved in a controlled limit in the case of Little String Theories (LSTs).

Starting with the first possibility the objective in this section is just give an approximate idea of how dual coodinates effects change the way these scales are derived with the non-doubled theory.

5.1 Large Volume

Supersymmetry broken at 1TeV stabilises the Higgs mass against radiative corrections generating the weak scale through dynamical electroweak symmetry breaking. In supergravity models the scale of supersymmetry breaking is set by the gravitino mass [11]:

$$m_{3/2} = e^{\mathcal{K}/2}\frac{|W|}{M_P^2} \ , \tag{74}$$

which will be obtained by working in the framework of N = 1 supergravity whose relevant terms in the action are:

$$S_{\mathrm{N}=1} = \int \left(\frac{M_P^2}{2}R - e^{\mathcal{K}}(\mathcal{G}^{a\bar{b}}D_a W D_{\bar{b}}\bar{W} - \frac{3}{M_P^2}W\bar{W}) \right) \sqrt{-g}d^4x \ , \tag{75}$$

where $\mathcal{K}$ and W are the Kähler potential and superpotential respectively.

Now, the idea is to reduce the double effective action and get a N=1 supergravity by compactification on a Calabi-Yau orientifold $\mathfrak{M}$ with holomorphic (3,0) form Ω and primitive (2,1) forms χ_α. In [12] some scales were derived from this kind of compactification. A similar approach will be taken here but including the factor coming from the dual space. The double action in this case is:

$$S = \frac{2\pi}{l_s^{18}} \int dx d\tilde{x} \left(e^{-2d}\, \mathcal{R}(\mathcal{H}, d) + \frac{1}{4}(\partial\!\!\!/\chi)^\dagger \ \mathbb{S} \ \partial\!\!\!/\chi \right) \ . \tag{76}$$

To obtain the bosonic type IIB supergravity action, the spinor χ_+ of positive chirality must be used. Using an $O(D,D)$ transformation to a frame without $\tilde{x}$ dependence gives:

$$S = \frac{2\pi\tilde{V}_4\tilde{V}_6}{l_s^{18}} \int d^{10}x\sqrt{-g} \left[e^{-2\phi}(R + 4(\partial\phi)^2 - \frac{1}{12}H^{ijk}H_{ijk}) + \mathcal{L}_{\mathrm{RR}} \right] \ , \tag{77}$$

in this way the lagrangian corresponding to the RR part of (76) reduces to kinetic terms:

$$\mathcal{L}_{\mathrm{RR}} = -\frac{F_1^2}{2} - \frac{\tilde{F}_3^2}{12} - \frac{\tilde{F}_5^2}{4\cdot 5!} \ , \tag{78}$$

where $\tilde{F}_3 = F_3 - C_0 H$ and $\tilde{F}_5 = F_5 - \frac{1}{2}C_2 \wedge H + \frac{1}{2}B \wedge F_3$ and the Chern-Simons contribution has been neglected. Following the conventions of [13] will be useful to define:

$$G_3 = F_3 - \tau H \ , \quad \tau = C_0 + ie^{-\phi} \ . \tag{79}$$

It will be also rewarding to work with $\tilde{\mathsf{V}}_4 = \tilde{V}_4/l_s^4$ and $\tilde{\mathsf{V}}_6 = \tilde{V}_6/l_s^6$. This gives:

$$S = \frac{2\pi\tilde{\mathsf{V}}_4\tilde{\mathsf{V}}_6}{l_s^8} \int d^{10}x\sqrt{-g} \left\{ e^{-2\phi}(R + 4\partial_j\phi\partial^j\phi) - \frac{F_1^2}{2} - \frac{G_3\cdot\bar{G}_3}{12} - \frac{\tilde{F}_5^2}{4\cdot 5!} \right\} \ . \tag{80}$$

Then it is convenient to perform a Weyl-rescaling $g_{ij} \to e^{\frac{\phi-\phi_0}{2}}g_{ij}$ to work in Einstein frame:

$$S = \frac{2\pi\tilde{\mathsf{V}}_4\tilde{\mathsf{V}}_6}{l_s^8} \int d^{10}x\sqrt{-g} \left\{ Re^{-2\phi_0} - e^{2(\phi-\phi_0)}\frac{\partial_j\tau\partial^j\bar{\tau}}{2} - \frac{e^{\phi-\phi_0}G_3\cdot\bar{G}_3}{12} - \frac{\tilde{F}_5^2}{4\cdot 5!} \right\} \ . \tag{81}$$

At large volume and in the absence of warping $\tilde{F}_5$ and $\partial_j \tau$ vanish, so reduction yields:

$$S = \frac{2\pi \tilde{\mathsf{V}}_4 \tilde{\mathsf{V}}_6}{g_s^2 l_s^8} \left\{ \int d^4x \sqrt{-g_4} R \mathcal{V} l_s^6 - \int d^4x \sqrt{-g_4} \left(\int_{\mathfrak{M}} d^6x \sqrt{g_6} e^{\phi_0 + \phi} \frac{G_3 \cdot \bar{G}_3}{12} \right) \right\} , \quad (82)$$

where $\mathcal{V} l_s^6 = \int \sqrt{g_6} d^6x$, comparing the first term with $S_{EH} = \frac{M_P^2}{2} \int R\sqrt{-g} d^4x$:

$$M_P^2 = \frac{4\pi \mathcal{V} \tilde{\mathsf{V}}_4 \tilde{\mathsf{V}}_6}{g_s^2 l_s^2} . \quad (83)$$

Using $\int \dfrac{G_3 \cdot \bar{G}_3}{3!} d^6x \sqrt{g_6} = \int G_3 \wedge *\bar{G}_3$ the second term, denoted by S_{G_3}, will lead

$$S_{G_3} = -\frac{2\pi \tilde{\mathsf{V}}_4 \tilde{\mathsf{V}}_6}{g_s l_s^8} \int d^4x \sqrt{-g_4} e^{\phi} \int_{\mathfrak{M}} \frac{G_3 \wedge *\bar{G}_3}{2} \quad (84)$$

Also $G_3 \wedge *\bar{G}_3 = \frac{1}{i} G_3 \wedge \bar{G}_3 + 2G_3^+ \wedge *\bar{G}_3^+$, where $G_3^+ = (G_3 + i * G_3)/2$. Then using a superpotential of the Gukov-Vafa-Witten form [14]:

$$W_0 = \frac{1}{l_s^2} \int_{\mathfrak{M}} G_3 \wedge \Omega , \quad (85)$$

the computation of the $G_3^+ \wedge *\bar{G}_3^+$ contribution is an usual calculation[3]:

$$\frac{1}{l_s^4} \int_{\mathfrak{M}} G_3^+ \wedge *\bar{G}_3^+ = \frac{i}{\varpi} (\mathcal{K}^{a\bar{b}} D_a W_0 D_{\bar{b}} \bar{W}_0 - 3 W_0 \bar{W}_0) , \quad (86)$$

where $\varpi = \int \Omega \wedge \bar{\Omega}$, therefore the action becomes:

$$S_{G_3} = -\frac{2\pi \tilde{\mathsf{V}}_4 \tilde{\mathsf{V}}_6}{g_s l_s^4} \int d^4x \sqrt{-g_4} \frac{e^{\phi}}{-i\varpi} (\mathcal{K}^{a\bar{b}} D_a W_0 D_{\bar{b}} \bar{W}_0 - 3 W_0 \bar{W}_0) , \quad (87)$$

the $1/l_s^4$ factor will be replaced by $\frac{M_P^4 g_s^4}{(4\pi \tilde{\mathsf{V}}_4 \tilde{\mathsf{V}}_6 \mathcal{V})^2}$, and $\frac{e^{\phi}}{2\mathcal{V}^2(-i\varpi)}$ will be arranged such a way that preserves the standard form of the Kähler potential:

$$\mathcal{K} = -2 \ln \mathcal{V} - \ln \left(-i \int_{\mathfrak{M}} \Omega \wedge \bar{\Omega} \right) - \ln(-i(\tau - \bar{\tau})) . \quad (88)$$

The S_{G_3} term takes the standard N = 1 form of (75):

$$S_{G_3} = -\int e^{\mathcal{K}} \left(\mathcal{G}^{a\bar{b}} D_a W D_{\bar{b}} \bar{W} - \frac{3}{M_P^2} W \bar{W} \right) \sqrt{-g_4} d^4x , \quad \mathcal{G}^{a\bar{b}} = \mathcal{K}^{a\bar{b}} / M_P^2 , \quad (89)$$

with

$$W = \frac{g_s^{\frac{3}{2}} M_P^3}{\sqrt{4\pi} \sqrt{\tilde{\mathsf{V}}_4 \tilde{\mathsf{V}}_6} l_s^2} \int_{\mathfrak{M}} G_3 \wedge \Omega \quad = \quad \frac{g_s^{\frac{3}{2}} M_P^3}{\sqrt{4\pi} \sqrt{\tilde{\mathsf{V}}_4 \tilde{\mathsf{V}}_6}} W_0 . \quad (90)$$

[3] The $G_3 \wedge \bar{G}_3$ term is topological and does not involve the relevant fields. Then the main idea is to expand G_3^+ in a basis of 3-forms: $G_3^+ = \frac{1}{\varpi}(\Omega \int G_3 \wedge \bar{\Omega} + \mathcal{K}^{\alpha\bar{\beta}} \bar{\chi}_{\bar{\beta}} \int G_3 \wedge \chi_\alpha)$ where $\mathcal{K}_{\alpha\bar{\beta}} = \frac{-1}{\varpi} \int \chi_\alpha \wedge \bar{\chi}_{\bar{\beta}}$. See [13, 15] for more details.

As usual, the complex structure contribution $e^{\mathcal{K}_{CS}} = (-i\varpi)^{-1}$ will be neglected. Therefore the gravitino mass is lowered by volume factors:

$$m_{3/2} = \frac{g_s^2|W_0|}{\sqrt{4\pi}\mathcal{V}\sqrt{\tilde{\mathsf{V}}_4\tilde{\mathsf{V}}_6}} M_P \ . \tag{91}$$

Volume factors that result in $m_{3/2} \sim$ TeV are required to explain the hierarchy between the weak scale and the Planck mass. However, the $\tilde{\mathsf{V}}_4$ factor seems problematic due to its extremally tiny value. If this can not be solved, a TeV mass will not be able to obtain in this way. Perhaps, what dual effects are bringing to light here is that there is no alternative than consider a small g_s coupling. The study of the properties of the dual volume $\tilde{\mathsf{V}}_6$ can help to find out if the small value of $\tilde{\mathsf{V}}_4$ can be compensated.

5.2 Tiny string coupling

LSTs are 6D strongly coupled non-Lagrangian theories generated by stacks of NS5 branes in type II theories. The LSTs are dual to[4] a local 7D theory with a linearly varying dilaton background in the infinite seventh dimension. The infinite extra dimension needs to be rendered finite with two branes and the two aditional transverse dimensions are compactified on a negligible scale. This gives rise to a 5D model introduced in [16] which is used to obtain a finite 4D Planck mass.

Starting with a DFT action the reduction to non-doubled physics according to that model should yield:

$$S = \frac{2\pi\tilde{\mathsf{V}}_4\tilde{\mathsf{V}}_6\mathsf{V}_5}{l_s^3}\int d^5x\sqrt{-g}e^{-2\phi}(R + 4(\partial\phi)^2 - \Lambda) \ , \tag{92}$$

with a 3-brane at $y = 0$ (hidden brane) and another 3-brane at $y = r_c$ (visible brane), DFT effects on the branes have been neglected completely[5].

Imposing a linearly varying dilaton background $2\phi = \alpha y$ the dual LST solution arises. The gravity field equations in this background are solved by the following bulk metric:

$$ds^2 = e^{-\frac{2}{3}\alpha y}(\eta_{\mu\nu}dx^\mu dx^\nu + dy^2) \ . \tag{93}$$

In order to obtain 4D physics from (92) it is not needed to rescale since ϕ depends only on the extra dimension y and can be viewed as a g_s. The dimensional reduction in this context is achieved by integrating over the extra dimension y, comparing with S_{EH}:

$$M_P^2 = \frac{4\pi\tilde{\mathsf{V}}_4\tilde{\mathsf{V}}_6\mathsf{V}_5}{l_s^3} 2\int_0^{r_c} e^{-\alpha y}dy = \frac{8\pi\tilde{\mathsf{V}}_4\tilde{\mathsf{V}}_6\mathsf{V}_5}{\alpha l_s^3}(1 - e^{-\alpha r_c}) \ . \tag{94}$$

[4] Holography allows to replace n-dimensional theories strongly coupled without gravity by weakly coupled theories of gravity embedded in n+1 dimensions.

[5] The setup of this model $S = S_{bulk} + S_{vis.brane} + S_{hid.brane}$ contain brane actions that have not been investigated yet in the context of DFT. However, one can realize that these effects do not contribute to the result since only the VEV derived from the action of the brane which contains the Standard Model fields is important but not the factors in the action.

So, the Planck scale is affected by the dual volumens.

The linear dilaton geometry provides an explanation for the weak/Planck hierarchy. Rewriting the metric in physical coordinates $dz = e^{-\frac{1}{3}\alpha y}dy$:

$$ds^2 = \left(1 - \frac{\alpha z}{3}\right)^2 \eta_{\mu\nu}dx^\mu dx^\nu + dz^2 \quad , \quad z = \frac{3}{\alpha}(1 - e^{-\frac{1}{3}\alpha y}) \ , \tag{95}$$

a comparison between induced metrics at branes can be done. Therefore the induced metrics are $g_{\mu\nu}(y = 0) = \eta_{\mu\nu}$ and $g_{\mu\nu}(y = r_c) = \left(1 - \frac{\alpha z_0}{3}\right)^2 \eta_{\mu\nu}$. These metrics are low-energy gravitational fields and couple to the 3-brane fields. The mass scales can be determined by redefining the fields to be canonically normalized. Consider the actions for the fundamental Higgs with vacuum expectation value (VEV) parameter v_0 and quartic self-coupling constant λ:

$$S_{vis(hid)} \supset \int d^4x \sqrt{-g_{vis(hid)}} \left\{-g^{\mu\nu}_{vis(hid)} D_\mu H^\dagger D_\nu H - \lambda^2(|H|^2 - v_0^2)^2\right\} \ . \tag{96}$$

Substituting the induced metrics into these actions yields $\left(1 - \frac{\alpha z_0}{3}\right)^2 = a_0$ factors in $S_{vis} \supset \int d^4x \sqrt{a_0^4} \left\{-a_0^{-1}\eta^{\mu\nu} D_\mu H^\dagger D_\nu H - \lambda^2(|H|^2 - v_0^2)^2\right\}$ in contrast with the simpler $S_{hid} \supset \int \left\{-\eta^{\mu\nu} D_\mu H^\dagger D_\nu H - \lambda^2(|H|^2 - v_0^2)^2\right\} d^4x$. After wave-function renormalization, $H \to (1 - \frac{\alpha z_0}{3})^{-1}H$, the standard form to analyse the VEV behaviour arises

$$S_{vis} \supset \int d^4x \left\{-\eta^{\mu\nu} D_\mu H^\dagger D_\nu H - \lambda^2(|H|^2 - a_0 v_0^2)^2\right\} \ . \tag{97}$$

The decreasing factor is an exponential law $v = e^{-\frac{1}{3}\alpha r_c} v_0$. The Higgs VEV is decreased on the visible brane relative to the hidden one so the weak mass scale is small compared to M_P. Here the weak scale is not affected directly by dual volumes.

6. Concluding remarks

The nature of the double spacetime is a subject currently in development. In this work double field theory symmetries have been reviewed from the double string point of view. In some sense the arising of Lorentz symmetry and T-duality coming both from the $O(D, D)$ group have been explained. The dual coordinates should not be viewed as artifacts, the doubled geometry is physical. In consequence, the dual dimensions should have effects on quantities calculated with string theory such as the weak scale. However, a further study is needed to reach a complete understanding of these effects and unleash the potential of these kind of theories.

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Canonical quantization of the Dirac oscillator field in (1+1) and (3+1) dimensions

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Abstract: The main goal of this work is to study the Dirac oscillator as a quantum field using the canonical formalism of quantum field theory and to develop the canonical quantization procedure for this system in $(1+1)$ and $(3+1)$ dimensions. This is possible because the Dirac oscillator is characterized by the absence of the Klein paradox and by the completeness of its eigenfunctions. We show that the Dirac oscillator field can be seen as constituted by infinite degrees of freedom which are identified as decoupled quantum linear harmonic oscillators. We observe that while for the free Dirac field the energy quanta of the infinite harmonic oscillators are the relativistic energies of free particles, for the Dirac oscillator field the quanta are the energies of relativistic linear harmonic oscillators.

Keywords: Quantum Field Theory; Dirac Oscillator; Canonical Quantization Formalism; Relativistic Harmonic Oscillators
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1. Introduction

Quantum states of a relativistic massive fermion are described by four-components wave functions called Dirac spinors. These wave functions, which are solutions of the Dirac equation, describe states of positive and negative energy. For the case where fermions carry the electric charge q, the electromagnetic interaction of fermions can be included by

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means of the electromagnetic fourpotential A^μ, which is introduced in the Dirac equation through the so called minimal substitution, changing the fourmomentum such as $p^\mu \to p^\mu - qA^\mu$. It is also possible to introduce a linear harmonic potential in the Dirac equation by substituting $\vec{p} \to \vec{p} - im\omega\beta\vec{r}$, where m is the fermion mass, ω represents an oscillator frequency, r is the distance of the fermion respects to the origin of the linear potential and $\beta = \gamma^0$ corresponds to the diagonal Dirac matrix.

The Dirac equation including the linear harmonic potential was initially studied by Itô et al. [1], Cook [2] and Ui et al. [3]. This system was latterly called by Moshinsky and Szczepaniak as Dirac oscillator [4], because it behaves as an harmonic oscillator with a strong spin-orbit coupling in the non-relativistic limit. As a relativistic quantum mechanical system, the Dirac oscillator has been widely studied. Several properties from this system have been considered in (1+1), (2+1), (3+1) dimensions [5]-[28]. Specifically, for the Dirac oscillator have been studied several properties as its covariance [6], its energy spectrum, its corresponding eigenfunctions and the form of the electromagnetic potential associated with its interaction in (3+1) dimensions [7], its Lie Algebra symmetries [8], the conditions for the existence of bound states [9], its connection with supersymmetric (non-relativistic) quantum mechanics [10], the absence of the Klein paradox in this system [11], its conformal invariance [12], its complete energy spectrum and its corresponding eigenfunctions in (2+1) dimensions [13], the existence of a physical picture for its interaction [14]. For this system, other aspects have been also studied as the completeness of its eigenfunctions in (1+1) and (3+1) dimensions [15], its thermodynamic properties in (1+1) dimensions [16], the characteristics of its two-point Green functions [17], its energy spectrum in the presence of the Aharonov-Bohm effect [18], the momenta representation of its exact solutions [19], the Lorenz deformed covariant algebra for the Dirac oscillator in (1+1) dimension [20], the properties of its propagator in (1+1) dimensions using the supersymmetric path integral formalism [21], its exact mapping onto a Jaynes-Cummings model [22], its nonrelativistic limit in (2+1) dimensions interpreted in terms of a Ramsey-interferometry effect [23], the existence of a chiral phase transition for this system in (2+1) dimensions in presence of a constant magnetic field [24], a new representation for its solutions using the Clifford algebra [25], its dynamics in presence of a two-component external field [26], the relativistic Landau levels for this system in presence of a external magnetic field in (2+1) dimensions [27] and its relationship with (Anti)-Jaynes-Cummings models in a (2+1) dimensional noncommutative space [28].

Some possible applications of the Dirac oscillator have been developed. For instance, the hadronic spectrum has been studied using the two-body Dirac oscillator in [29, 5] and the references therein. The Dirac oscillator in (2+1) dimension has been used as a framework to study some condensed matter physical phenomena such as the study of electrons in two dimensional materials, which can be applied to study some aspects of the physics of graphene [30]. This system has also been used in quantum optics to describe the interaction of atoms with electromagnetic fields in cavities (the Jaynes-Cummings model) [30].

The standard point of view of the Quantum Field Theory (QFT) establishes that

an excitation of one of the infinite degrees of freedom that constitute the free Dirac quantum field can be interpreted as a free relativistic massive fermion [31]. The free Dirac quantum field which describes the quantum dynamics of a non-interacting relativistic massive fermion can be seen as constituted by infinite decoupled quantum harmonic oscillators. In this scheme a non-interacting relativistic massive fermion is described as an excitation of a degree of freedom of the free Dirac quantum field [31]. For the free Dirac field the energy quanta of the infinite harmonic oscillators are relativistic energies of free fermions or antifermions. An analogous situation is presented in the description of a non-interacting relativistic boson as an excitation of one of the infinite degrees of freedom (harmonic oscillators) that constitute a free bosonic quantum field [31]. For the free Boson field the energy quanta of the infinite harmonic oscillators are relativistic energies of free bosons.

On the other hand, the Dirac oscillator, which describes the interacting system constituted by a relativistic massive fermion under the action of a linear harmonic potential, has not been studied as a quantum field up to now. In this direction, we show that it is possible to study consistently the Dirac oscillator as a QFT system. The main goal of this work is to consider the Dirac oscillator as a field and perform the canonical quantization procedure of the Dirac oscillator field in (1+1) and (3+1) dimensions. We can do it, in the first place, because the Dirac oscillator is characterized by the absence of the Klein paradox [11], that allows to distinguish between positive and negative energy states [32]. In the second place, the Dirac oscillator is an interacting system in which there exist bound states that are defined by means of specific quantum numbers. This aspect means that the solution of the Dirac oscillator equation leads to both a well-known energy spectrum and an eigenfunction set which satisfies the completeness and orthonormality conditions in (1+1) and (3+1) dimensions [15]. Thus the Dirac oscillator field can be consistently treated as a quantum field because it can be written in terms of a Fourier expansion of that eigenfunction set. We observe that after the canonical quantization procedure, the Dirac oscillator field can be seen as constituted by infinite relativistic decoupled quantum linear harmonic oscillators.

The canonical quantization procedure for the Dirac oscillator field starts from writing the Hamiltonian operator for this system in terms of annihilation and creation operators satisfying the usual anticommutation relations. This procedure allow us to obtain the Feynman propagator for the Dirac oscillator field. Thus we are able to study the differences between the Dirac oscillator field and the free Dirac field. We note that while for the free Dirac field the energy quanta of the infinite harmonic oscillators are relativistic energies of free particles, for the Dirac oscillator quantum field the quanta are energies of relativistic linear harmonic oscillators.

This work is presented as follows: First, in section 2 we introduce the notation used in this paper presenting the relevant aspects of the standard canonical quantization procedure for the free Dirac field in 3+1 dimensions. Next, in section 3 we study the Dirac oscillator system in 1+1 dimensions from a point of view of quantum relativistic mechanics; additionally we describe some aspects of the Dirac's sea picture for this system and

we develop the canonical quantization procedure for the Dirac oscillator field in 1+1 dimensions. Posteriorly, in section 4 we study the canonical quantization procedure for the Dirac oscillator field in 3+1 dimensions. Finally, in section 5 we present some conclusions of this work.

2. Canonical quantization for the free Dirac field

In this section we present the main steps of the standard canonical quantization procedure of the free Dirac field following the procedure developed in [31]. The equation of motion for a free relativistic fermion of mass m is given by the Dirac equation

$$(i\gamma^\mu\partial_\mu - m)\psi(\vec{r},t) = 0, \tag{1}$$

where we have taken $\hbar = c = 1$. In this equation γ^μ represents the Dirac matrices obeying the Clifford algebra $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$ and $\psi(\vec{r},t)$ represents the four-component spinor wave functions. The four linearly independent solutions from the Dirac equation are plane waves of the form $\psi^r_{\vec{p}}(\vec{r},t) = (2\pi)^{-3/2}\sqrt{\frac{m}{E_{\vec{p}}}}w_r(\vec{p})e^{-i\epsilon_r(E_{\vec{p}}t-\vec{p}\cdot\vec{r})}$, where $E_{\vec{p}}$ is given by $E_{\vec{p}} = +\sqrt{\vec{p}^{\,2}+m^2}$. The two solutions of relativistic free-particle positive energy are described by $r = 1,2$ while the two solutions of relativistic free-particle negative energy are described by $r = 3,4$. The sign function ϵ_r takes the value $\epsilon_r = 1$, for $r = 1,2$, and $\epsilon_r = -1$, for $r = 3,4$. Additionally, the spinors $w_r(\vec{p})$ obey the equation $(\gamma_\mu p^\mu - \epsilon_r m)w_r(\vec{p}) = 0$.

In a quantum field theory treatment, $\psi(\vec{r},t)$ is the free Dirac field. The Hamiltonian associated to this field is $H = \int d^3x\psi^\dagger(-i\boldsymbol{\alpha}\cdot\nabla + m\beta)\psi$, where the matrices $\boldsymbol{\alpha}, \beta$ are defined as $\alpha_i = \gamma^0\gamma^i$, $\beta = \gamma^0$. The canonical quantization procedure of the free Dirac field starts by replacing the fields $\psi(\vec{r},t)$ and $\psi^\dagger(\vec{r},t)$ with the fields operators $\hat{\psi}(\vec{r},t)$ and $\hat{\psi}^\dagger(\vec{r},t)$ which obey the usual commutation relations of the Jordan-Wigner type [31]. The Fourier expansion of the free Dirac field operator $\hat{\psi}(\vec{r},t)$, in terms of the plane waves functions, is written as

$$\hat{\psi}(\vec{r},t) = \sum_{r=1}^{4}\int \frac{d^3p}{(2\pi)^{3/2}}\sqrt{\frac{m}{E_{\vec{p}}}}\hat{a}(\vec{p},r)w_r(\vec{p})e^{-i\epsilon_r p\cdot x}, \tag{2}$$

where $p\cdot x \equiv p_\mu x^\mu = E_{\vec{p}}t - \vec{p}\cdot\vec{r}$ and $\hat{a}(\vec{p},r)$ is an operator. The operators $\hat{a}(\vec{p},r)$ and its conjugated $\hat{a}^\dagger(\vec{p},r)$ satisfy the usual anticommutation relations. The Hamiltonian operator $\hat{H}$ associated to this system is defined in terms of the Dirac field operators $\hat{\psi}(\vec{r},t)$ and $\hat{\psi}^\dagger(\vec{r},t)$. Using the Fourier expansion of the free Dirac field operator, $\hat{H}$ can be written as $\hat{H} = \int d^3pE_{\vec{p}}\left[\sum_{r=1}^{2}\hat{n}_{\vec{p},\vec{r}} + \sum_{r=3}^{4}\hat{\bar{n}}_{\vec{p},\vec{r}}\right]$, where $\hat{n}_{\vec{p},\vec{r}} = \hat{a}^\dagger(\vec{p},r)\hat{a}(\vec{p},r)$ is the particle-number operator and $\hat{\bar{n}}_{\vec{p},\vec{r}} = \hat{a}(\vec{p},r)\hat{a}^\dagger(\vec{p},r)$ is the antiparticle-number operator. In this Hamiltonian operator, which is positively defined, the Dirac's see picture has been used and the zero-point energy contribution has been subtracted [31].

Now it is possible to introduced the following canonical transformations over the annihilation and creation operators that allow to differentiate between operators associated

to particles and antiparticles: $\hat{c}(\vec{p},+s) = \hat{a}(\vec{p},1)$, $\hat{c}(\vec{p},-s) = \hat{a}(\vec{p},2)$, $\hat{d}^\dagger(\vec{p},+s) = \hat{a}(\vec{p},3)$, $\hat{d}^\dagger(\vec{p},-s) = \hat{a}(\vec{p},4)$. With these transformations, the following identifications are possible: $\hat{c}(\vec{p},s)$ and $\hat{c}^\dagger(\vec{p},s)$ represent the annihilation and the creation operators of particles, respectively; $\hat{d}(\vec{p},s)$ and $\hat{d}^\dagger(\vec{p},s)$ represent the annihilation and the creation operators of antiparticles, respectively. Using the operators $\hat{c}(\vec{p},s)$, $\hat{c}^\dagger(\vec{p},s)$, $\hat{d}(\vec{p},s)$ and $\hat{d}^\dagger(\vec{p},s)$, the Hamiltonian operator of the free Dirac field can be written as [31]

$$\hat{H} = \sum_s \int d^3p\, E_{\vec{p}} \left(\hat{n}^{(c)}_{\vec{p},s} + \hat{n}^{(d)}_{\vec{p},s} \right), \tag{3}$$

where $\hat{n}^{(c)}_{\vec{p},s} = \hat{c}^\dagger(\vec{p},s)\hat{c}(\vec{p},s)$ represents the fermion-number operator and $\hat{n}^{(d)}_{\vec{p},s} = \hat{d}^\dagger(\vec{p},s)\hat{d}(\vec{p},s)$ represents the antifermion-number operator. It is possible to observe that $\hat{c}^\dagger(\vec{p},s)$ creates a fermion of mass m having energy $E_{\vec{p}}$, momentum $\vec{p}$, charge $+e$ and projection of spin $+s$, whereas $\hat{d}^\dagger(\vec{p},s)$ creates an antifermion of the same mass having the same energy, the same momentum, charge $-e$ and projection of spin $-s$. In this scheme, a free relativistic massive fermion is described as an excitation from a degree of freedom of the free Dirac quantum field. For the free Dirac field, the energy quanta of the infinite harmonic oscillators are the relativistic energies $E_{\vec{p}}$ of free fermions or antifermions.

Finally, the Feynman propagator $\mathcal{S}^F_{\alpha\beta}(x-y)$ of the free Dirac field is defined as $i\mathcal{S}^F_{\alpha\beta}(x-y) = \langle 0|\, \hat{T}\left(\hat{\psi}_\alpha(x)\hat{\bar{\psi}}_\beta(y) \right) |0\rangle$. By using the plane wave expansion, this propagator can be re-written as $i\mathcal{S}^F_{\alpha\beta}(x-y) = (i\gamma^\mu\partial_\mu - m)\delta_{\alpha\beta} i\Delta_F(x-y)$, where $i\Delta_F(x-y)$ represents the Feynman propagator of a scalar field.

3. Dirac oscillator in the (1+1) dimensional case

One of the most important and well studied systems in non-relativistic quantum mechanics is the harmonic oscillator. This system is characterized by its simplicity and usefulness. The harmonic oscillator is a simple system due to the fact that its Hamiltonian operator is quadratic both in coordinates and momenta. An analogous system in relativistic quantum mechanics is given by a linear interaction term introduced in the Dirac equation. In fact the Dirac equation corresponds to the linearization of relativistic Schrödinger equation. So the harmonic potential must be introduced in the Dirac equation as the quadratic root of the quadratic potential, i. e. the potential must be linear. The interacting system constituted by a relativistic massive fermion under the action of a linear potential is known as the Dirac oscillator [4]-[29].

In this section we will consider the Dirac oscillator in the (1+1) dimensional case. Initially we will obtain the energy spectrum and the wave functions describing the quantum states of this system. Next we develop the canonical quantization procedure for the Dirac oscillator field in 1+1 dimensions.

3.1 Spectrum and wave functions in (1+1) dimensions

The linear interaction over a relativistic fermion of mass m that moves in one-dimension over the z axis is introduced into the Dirac equation [4, 7, 14] by substituting the momentum p_z as $\hat{p}_z \to \hat{p}_z - im\omega\gamma^0\hat{z}$, where ω is the frequency associated to the oscillator. Thus, the one-dimensional Dirac oscillator equation takes the form ($\hbar = c = 1$)

$$i\frac{\partial}{\partial t}|\psi\rangle = (\alpha_3 \cdot (\hat{p}_z - im\omega\beta\hat{z}) + \beta m)|\psi\rangle . \tag{4}$$

The solution of equation (4) allows to obtain both the energy spectrum and the wave functions describing the quantum states of this system. As we will show furtherly, these wave functions will be used in the canonical quantization procedure of this system. Given that the interaction does not mix positive and negative energies, it can be possible to rewrite the four-component spinor state in eq. (4) as

$$|\psi\rangle = \begin{pmatrix} |\phi\rangle \\ |\chi\rangle \end{pmatrix}, \tag{5}$$

where $|\phi\rangle$ and $|\chi\rangle$ are spinors. If we use the standard representation of the Dirac matrices, the time independent Hamiltonian equation takes the form

$$\begin{pmatrix} m & \sigma_3 \cdot (\hat{p}_z + im\omega\hat{z}) \\ \sigma_3 \cdot (\hat{p}_z - im\omega\hat{z}) & -m \end{pmatrix} \begin{pmatrix} |\phi\rangle \\ |\chi\rangle \end{pmatrix} = E \begin{pmatrix} |\phi\rangle \\ |\chi\rangle , \end{pmatrix} \tag{6}$$

which leads to the following coupled equations

$$(E - m)|\phi\rangle = \sigma_3 \cdot (\hat{p}_z + im\omega\hat{z})|\chi\rangle , \tag{7a}$$
$$(E + m)|\chi\rangle = \sigma_3 \cdot (\hat{p}_z - im\omega\hat{z})|\phi\rangle . \tag{7b}$$

Starting from (7a) and (7b), we obtain that

$$\left(\hat{p}_z^2 + (m\omega)^2\hat{z}^2\right)|\psi\rangle = \left((E^2 - m^2)\mathbb{I} + m\omega\beta\right)|\psi\rangle , \tag{8}$$

where the commutation relations between the operators $\hat{p}_z$ and $\hat{z}$ have been taken into account. In the last equation, $\mathbb{I}$ is the identity matrix 4×4 and β is the Dirac matrix γ^0. Using the following reparametrization [7]

$$\hat{\zeta} = (m\omega)^{\frac{1}{2}}\hat{z}, \tag{9a}$$
$$\hat{p}_\zeta = (m\omega)^{-\frac{1}{2}}\hat{p}_z, \tag{9b}$$

it is possible to write the last equation as

$$\left(\hat{p}_\zeta^2 + \hat{\zeta}^2\right)|\psi\rangle = \eta_\pm|\psi\rangle , \tag{10}$$

where

$$\eta_{\pm} = \frac{E^2 - m^2}{m\omega} \pm 1. \tag{11}$$

We note that the equations (7a) and (7b), for the states $|\phi\rangle$ and $|\chi\rangle$, allow to the equation (10) that has the form of a harmonic oscillator equation. This result suggests us that we can introduce the creation and annihilation operators given by

$$\hat{a}^{\dagger} = \frac{1}{\sqrt{2}}\left(\hat{\zeta} - i\hat{p}_{\zeta}\right), \quad \hat{a} = \frac{1}{\sqrt{2}}\left(\hat{\zeta} + i\hat{p}_{\zeta}\right), \tag{12}$$

with the purpose to obtain the eigenvalues of the system. In this way, using these operators, the equations (7a) and (7b) have the form

$$|\phi\rangle = i\frac{\sqrt{2m\omega}}{E-m}\sigma_3\hat{a}^{\dagger}\,|\chi\rangle\,, \tag{13a}$$

$$|\chi\rangle = -i\frac{\sqrt{2m\omega}}{E+m}\sigma_3\hat{a}\,|\phi\rangle\,. \tag{13b}$$

Substituting again (13a) into (13b), it is possible to find the following equation for the state $|\phi\rangle$

$$|\phi\rangle = \frac{2m\omega}{E_n^2 - m^2}\hat{N}\,|\phi\rangle\,, \tag{14}$$

while the state $|\chi\rangle$ satisfies

$$|\chi\rangle = \frac{2m\omega}{E_{n'+1}^2 - m^2}(\hat{N}+1)\,|\chi\rangle\,, \tag{15}$$

where $\hat{N} = \hat{a}^{\dagger}\hat{a}$ is the occupation number operator of the state. Since $\hat{N}$ determines the energy level of the state where the particle (antiparticle) is, then the energy spectrum can be deduced from (14) and (15). Therefore we have

- From (13a), the energy spectrum for positive energy states is

$$E_n^2 = 2nm\omega + m^2. \tag{16}$$

- From (13b), the energy spectrum for negative energy states is

$$E_{n'+1}^2 = 2(n'+1)m\omega + m^2. \tag{17}$$

These energy spectrums for fermions and antifermions can be written simultaneously if we impose the following quantum number condition

$$n' = n - 1, \qquad \text{with } n \neq 0, \tag{18}$$

thus the total spectrum can be written as [15]

$$E_n = \pm\left(2\,|n|\,m\omega + m^2\right)^{\frac{1}{2}}, \qquad \text{con } n \in \mathbb{Z}. \tag{19}$$

The upper sign in this expression is taken for $n \geq 0$, while the lower sign is for $n < 0$ [15], therefore $E_{-n} = -E_n$ (for $n \neq 0$), i. e. the negative quantum numbers correspond to the negative energy states. We observe from (19), that the lower positive energy state whose energy value is m corresponds to the state with $n = 0$, while the greater negative energy state whose energy value is $-\left(2m\omega + m^2\right)^{\frac{1}{2}}$ corresponds to the state $n = -1$. Additionally, we observe that if $\omega \ll m$, then the energy difference between the states ϕ_0 and χ_{-1} is $\Delta E = 2m + \omega$. For $\omega = 0$, i. e. the harmonic potential vanishes into eq. (4), then $\Delta E = 2m$, which is a well known result obtained from the Dirac equation in the free case.

By using the previous results, the states of the system can be written as

$$|\psi_n\rangle = \begin{pmatrix} |\phi_n\rangle \\ |\chi_n\rangle \end{pmatrix}, \tag{20}$$

where the quantum number n, which is an integer number, can describe positive and negative energy states. Using the expression (13b), we can write that

$$|\psi_n\rangle = \begin{pmatrix} |\phi_n\rangle \\ -i\frac{\sqrt{2m\omega}}{E+m}\sigma_3 \hat{a} \, |\phi_n\rangle \end{pmatrix}. \tag{21}$$

If we apply the occupation number operator $\hat{N}$ over the state of a system described by $|\psi_n\rangle$, we obtain

$$\hat{N} \, |\psi_n\rangle = \begin{pmatrix} |n| \, |\phi_n\rangle \\ (|n| - 1) \, |\chi_n\rangle \end{pmatrix}, \tag{22}$$

where we have assumed that the state $|\phi_n\rangle$ has an occupation number given by $|n|$ and where we have used the relation (13b) and the properties of the creation and annihilation operators. For the last expression, we realize that the lowest spinor $|\chi_n\rangle$ has associated the occupation number given by $|n| - 1$. Thus the states $|\phi_n\rangle$ and $|\chi_n\rangle$ can be written as

$$|\phi_n\rangle = |n\rangle \, \xi_n^1, \tag{23a}$$

$$|\chi_n\rangle = |n - 1\rangle \, \xi_n^2, \tag{23b}$$

where $\xi_n^{1,2}$ represent the two-component spinors and $|n\rangle$ represents a state with occupation number $|n|$. In consequence, the states of the system are rewritten as

$$|\psi_n\rangle = \begin{pmatrix} |n\rangle \, \xi_n^1 \\ |n - 1\rangle \, \xi_n^2 \end{pmatrix}. \tag{24}$$

Taking into account that the creation $\hat{a}^\dagger$ and annihilation $\hat{a}$ operators satisfy that $\hat{a}^\dagger |n\rangle = \sqrt{|n|+1}\ \ |n+1\rangle$, $\hat{a}|n\rangle = \sqrt{|n|}\ \ |n-1\rangle$, then these operators acting on the state $|\psi_n\rangle$ allow to

$$\hat{a}^\dagger |\psi_n\rangle = \begin{pmatrix} \sqrt{|n|+1}\ |n+1\rangle\, \xi^1_{n+1} \\ \sqrt{|n|}\ |n\rangle\, \xi^2_{n+1} \end{pmatrix}, \text{ for } n \neq -1, \tag{25a}$$

$$\hat{a} |\psi_n\rangle = \begin{pmatrix} \sqrt{|n|}\ |n-1\rangle\, \xi^1_{n-1} \\ \sqrt{|n-1|}\ |n-2\rangle\, \xi^2_{n-1} \end{pmatrix}, \text{ for } n \neq 0, \tag{25b}$$

whereas these operators acting on the states $|\psi_0\rangle$ y $|\psi_{-1}\rangle$, which have associate the occupation numbers $n = 0, -1$, respectively, allow to

$$\hat{a} |\psi_0\rangle = \frac{1}{2}(1-\beta)|\psi_{-1}\rangle, \tag{26a}$$

$$\hat{a}^\dagger |\psi_{-1}\rangle = \sqrt{2}\,|\psi_0\rangle. \tag{26b}$$

In figure 1 we have schematically represented the action of the creation and annihilation operators on the positive and negative energy states. We observe that the effect of the annihilation operator on the state $|\psi_0\rangle$, which corresponds to the lowest positive energy state, is such that it does not annihilate the state but it drives it to the state $|\psi_{-1}\rangle$, which corresponds to the greater negative energy state. Likewise, the effect of applying the creation operator on the state $|\psi_{-1}\rangle$ is such that it does not annihilate that state, but drives it to the state $|\psi_0\rangle$. Therefore, we observe that the appearing of the negative energy states generates the well known problem of the Dirac theory: a minimal energy state does not exist, then it is possible to obtain an infinite energy amount from this system. In order to give a solution to this problem, it is necessary to introduce the Dirac's sea picture for the Dirac oscillator which will be performed by means of the canonical quantization for this system.

After calculating the energy spectrum of the one-dimensional Dirac oscillator, we proceed to obtain the wave functions. We substitute (5) into (10), then we obtain the following differential equation for the wave function associated to the bispinor $|\phi\rangle$

$$\left[\frac{d^2}{d\zeta^2} + \left(\eta_+ - \zeta^2\right)\right]\phi(\zeta) = 0, \tag{27}$$

where we have used the coordinate representation of the wave function given by $\phi(\zeta) = \langle\zeta|\,|\phi\rangle$. The differential equation (27) corresponds to the one of a relativistic harmonic oscillator, whose solution is [15]

$$\phi_n(\zeta) = N_{|n|} H_{|n|}(\zeta) e^{-\frac{\zeta^2}{2}} \xi^1_n. \tag{28}$$

Likewise the solution to the differential equation associated to the wave function $\chi_n(\zeta) = \langle\zeta|\,|\chi_n\rangle$ has the form

$$\chi_n(\zeta) = N_{|n|-1} H_{|n|-1}(\zeta) e^{-\frac{\zeta^2}{2}} \xi^2_n. \tag{29}$$

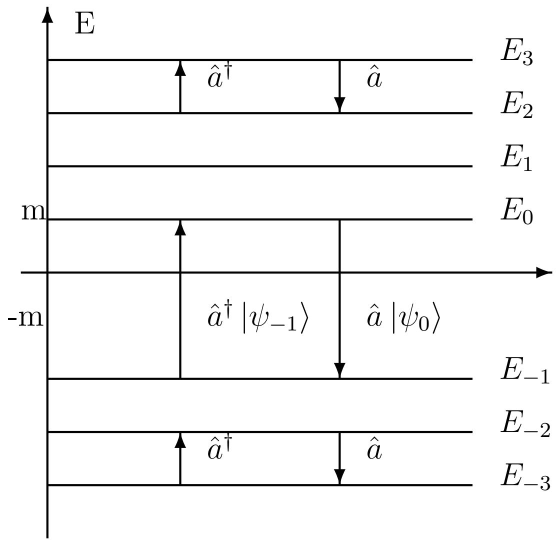

Figure 1 Energy spectrum of the one-dimensional Dirac oscillator and the action of the creation and annihilation operator on some system states.

In the expressions for $\phi_n(\zeta)$ and $\chi_n(\zeta)$, $H_n(\zeta)$ represent the Hermite polynomials. Now, from the expression (13b) we can obtain the following relation between the spinors ξ_n^1 and ξ_n^2

$$\xi_n^2 = -i\sqrt{\frac{E_n - m}{E_n + m}}\sigma_3\ \xi_n^1, \tag{30}$$

where we have used the properties of the creation and annihilation operators and the definitions given by (23). Now it is possible to write

$$\xi_n^1 = \begin{pmatrix} \sqrt{\frac{E_n+m}{2E_n}} \\ 0 \end{pmatrix}, \tag{31a}$$

$$\xi_n^2 = \begin{pmatrix} -i\sqrt{\frac{E_n-m}{2E_n}} \\ 0 \end{pmatrix}. \tag{31b}$$

We observe that the spinor ξ_n^2 is annihilated for the case $n = 0$, which implies $|\chi_0\rangle \equiv 0$. So the most general solution for the one-dimensional Dirac oscillator equation (4) is given by [15, 21]

$$\psi_n(z,t) = \sqrt{m\omega} \begin{pmatrix} \phi_n(z)\xi_n^1 \\ \chi_n(z)\xi_n^2 \end{pmatrix} e^{-iE_nt}, \tag{32}$$

where the normalization for the wave functions has been performed and $n = 0, \pm1, \pm2, \ldots$. Finally, we can write the one-dimensional Dirac oscillator equation (4) in its explicit covariant form [7, 14]

$$(i\gamma^\mu\partial_\mu - m + \sigma^{\mu\nu}F_{\mu\nu})\psi = 0, \tag{33}$$

where $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^{\mu}, \gamma^{\nu}]$ and the field-strength tensor $F_{\mu\nu}$ associated to the harmonic interaction presents in this system is given by

$$F_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu}, \tag{34}$$

where we have defined the fourpotential associated to the interaction as

$$A_{\mu} = \frac{1}{4}\left(2(u \cdot x)x_{\mu} - x^2 u_{\mu}\right). \tag{35}$$

In the last expression $u_{\mu} = (m\omega, \vec{0})$ is a fourvector that depends on the reference frame. In the references [7, 14], two different physical pictures for the interaction of the Dirac oscillator have been considered.

3.2 Canonical quantization procedure in (1+1) dimensions

By means of the previously defined quantities, we introduce the Lagrangian density of the system as

$$\mathcal{L} = \bar{\psi}(i\gamma^{\mu}\partial_{\mu} - m)\psi + \bar{\psi}\sigma^{\mu\nu}\psi\, F_{\mu\nu}, \tag{36}$$

which, by using the Euler-Lagrange motion equation, gives the one-dimensional Dirac oscillator equation (33). As the Lagrangian density has an explicit dependence on the position coordinate x, it is not possible to determine the energy-momentum tensor of the system [33]. However, it can be found that the Hamiltonian density is

$$\mathcal{H} = \psi^{\dagger}(-i\alpha_3 \cdot (\partial_z + m\omega\beta z) + \beta m)\psi. \tag{37}$$

To follow the standard canonical quantization procedure [31], we initially impose the following commutation relations of the Jordan-Wigner type for the fermion fields

$$\left\{\hat{\psi}_{\alpha}(z,t), \hat{\psi}^{\dagger}_{\beta}(z',t)\right\} = \delta_{\alpha\beta}\delta(z - z'), \tag{38}$$

$$\left\{\hat{\psi}_{\alpha}(z,t), \hat{\psi}_{\beta}(z',t)\right\} = 0, \tag{39}$$

$$\left\{\hat{\psi}^{\dagger}_{\alpha}(z,t), \hat{\psi}^{\dagger}_{\beta}(z',t)\right\} = 0. \tag{40}$$

Using these relations, we obtain the Hamiltonian operator for the Dirac oscillator field operator $\hat{\psi}$ in the form

$$\hat{H} = \int dz\, \hat{\psi}^{\dagger}(-i\alpha_3 \cdot (\partial_z + m\omega\beta z) + \beta m)\hat{\psi}. \tag{41}$$

Considering that the Heisenberg equation for the field operator $\hat{\psi}$ is given by $\dot{\hat{\psi}}(z,t) = -i[\hat{\psi}(z,t), \hat{H}]$, then we can obtain that the motion equation for the Dirac oscillator field operator is

$$i\dot{\hat{\psi}}(z,t) = (-i\alpha_3 \cdot (\partial_z + m\omega\beta z) + \beta m)\,\hat{\psi}(z,t). \tag{42}$$

Now we write the Dirac oscillator field operator using the wave functions of the Dirac oscillator (32) as the base of the expansion. These wave functions are written in terms of the Hermite polynomials which represent a complete set of orthonormal polynomials. Thus, the Fourier serie expansion for the Dirac oscillator field operator can be written as

$$\begin{aligned}\hat{\psi}(z,t) &= \sum_{n=-\infty}^{\infty} \hat{b}_n \psi_n(z,t) e^{-iE_n t} \\ &= \sum_{n=0}^{\infty} \hat{b}_n u_n(z) e^{-iE_n t} + \sum_{n=1}^{\infty} \hat{b}_{-n} \nu_n(z) e^{iE_n t} \\ &= \hat{\psi}_+(z,t) + \hat{\psi}_-(z,t), \end{aligned} \tag{43}$$

where the positive and negative energy contributions have been separated $\hat{\psi}_\pm(z,t)$ and the spinors $u_n(z)$ and $\nu_n(z)$ have been defined as

$$u_n(z) = \begin{pmatrix} \phi_n(z)\xi_n^1 \\ \chi_n(z)\xi_n^2 \end{pmatrix}, \ \nu_n(z) = \begin{pmatrix} \phi_{-n}(z)\xi_{-n}^1 \\ \chi_{-n}(z)\xi_{-n}^1 \end{pmatrix}. \tag{44}$$

If we make use of the anticommutation relations for the fields (38), (39) and (40), we can verify that creation $\hat{b}_n^\dagger$ and annihilation $\hat{b}_n$ operators of positive (for $n \geq 0$) and negative (for $n < 0$) energy particles satisfy the following anticommutation relations

$$\left\{\hat{b}_n, \hat{b}_m^\dagger\right\} = \delta_{n,m}, \tag{45}$$

$$\left\{\hat{b}_n, \hat{b}_m\right\} = 0, \tag{46}$$

$$\left\{\hat{b}_n^\dagger, \hat{b}_m^\dagger\right\} = 0. \tag{47}$$

Using these anticommutation relations, we can find that the Hamiltonian operator is now given by

$$\begin{aligned}\hat{H} &= \sum_{n=-\infty}^{\infty} E_n \hat{b}_n^\dagger \hat{b}_n \\ &= \sum_{n=0}^{\infty} E_n \hat{b}_n^\dagger \hat{b}_n - \sum_{n=1}^{\infty} E_n \hat{b}_{-n}^\dagger \hat{b}_{-n}, \end{aligned} \tag{48}$$

where we have also used the expansion of the field operator (43) and the properties of the Hermite polynomials. We can observe that the eigenvalues of this Hamiltonian operator can take negative values without restriction at all, because it is possible the creation of negative energy particles. To solve this problem we take the Dirac's sea picture for this system, so we impose that all the negative energy states are occupied by negative energy particles. This configuration is called the vacuum state of the Dirac oscillator field $|0\rangle$ that is written as [34]

$$|0\rangle = \prod_{n=1}^{\infty} \hat{b}_{-n}^\dagger \left|0_D\right\rangle, \tag{49}$$

where $|0_D\rangle$ is the Dirac vacuum state which is characterized for not been occupied by fermions of positive or negative energy. We observe that if we apply the creation operator of negative energy particle $\hat{b}^\dagger_{-m}|0\rangle$ on the vacuum state $|0\rangle$ we obtain

$$\hat{b}^\dagger_{-m}|0\rangle = \prod_{n=1}^{\infty} \hat{b}^\dagger_{-m}\hat{b}^\dagger_{-n}|0_D\rangle = 0, \tag{50}$$

which implies that it is not possible to create a new negative energy fermion because all the negative energy states are occupied. On the other hand, it is possible that a negative energy state can be annihilated and then a hole can be originated in the Dirac's sea. This hole represents an antiparticle with negative energy. If we use the anticommutation relations (45), (46) and (47) into (48), we find that the Hamiltonian operator takes the form

$$\hat{H} = \sum_{n=0}^{\infty} E_n \hat{b}^\dagger_n \hat{b}_n + \sum_{n=1}^{\infty} E_n \left(\hat{b}_{-n}\hat{b}^\dagger_{-n} - 1\right). \tag{51}$$

If the divergent energy of the vacuum $\sum_{n=1}^{\infty} E_n$ is subtracted, the Hamiltonian operator can be reduced to

$$\hat{H} = \sum_{n=0}^{\infty} E_n \hat{b}^\dagger_n \hat{b}_n + \sum_{n=1}^{\infty} E_n \hat{b}_{-n}\hat{b}^\dagger_{-n}. \tag{52}$$

We note that now this Hamiltonian operator does not have the problem observed before because it has been positively defined. Finally, we perform the following canonical transformations to clearly differentiate between the operators associated to particles and antiparticles [31]

$$\hat{b}^\dagger_n = \hat{c}^\dagger_n, \tag{53a}$$
$$\hat{b}_n = \hat{c}_n, \tag{53b}$$
$$\hat{b}_{-n} = \hat{d}^\dagger_n, \tag{53c}$$
$$\hat{b}^\dagger_{-n} = \hat{d}_n, \tag{53d}$$

where it is possible to identify $\hat{d}_n$ and $\hat{d}^\dagger_n$ as the annihilation and the creation operators of antiparticles, respectively. These operators satisfy the anticommutation relation $\left\{\hat{d}_n, \hat{d}^\dagger_m\right\} = \delta_{n,m}$. These transformations imply that now the application of the annihilation operator of negative energy particle on the vacuum state can be understood as the application of a creation operator of antiparticle. Using this notation, the Fourier serie expansion for the Dirac oscillator field (43) has now the form

$$\hat{\psi}(z,t) = \sum_{n=0}^{\infty} \hat{c}_n u_n(z) e^{-iE_n t} + \sum_{n=1}^{\infty} \hat{d}^\dagger_n v_n(z) e^{iE_n t}. \tag{54}$$

Now the Hamiltonian operator for the Dirac oscillator field can be written as

$$\hat{H} = \sum_{n=0}^{\infty} E_n \hat{c}^\dagger_n \hat{c}_n + \sum_{n=1}^{\infty} E_n \hat{d}^\dagger_n \hat{d}_n = \sum_{n=0}^{\infty} E_n \hat{n}^{(c)}_n + \sum_{n=1}^{\infty} E_n \hat{n}^{(d)}_n, \tag{55}$$

where $\hat{n}_n^{(c)}$ represents the number operator for fermions and $\hat{n}_n^{(d)}$ represents the number operator for antifermions. It is possible to observe that $\hat{c}_n^\dagger$ acting on $|0\rangle$ creates a fermion having energy E_n and charge $+e$, whereas $\hat{d}_n^\dagger$ creates antifermion having energy E_n and charge $-e$. In this scheme, an interacting relativistic massive fermion is described as an excitation from a degree of freedom of the Dirac oscillator quantum field. For the Dirac oscillator field the energy quanta E_n are the energies of linear harmonic oscillators. We observe that, in this sense, the Hamiltonian operator (55) is analogous to the Hamiltonian operator for the free Dirac field in three dimensions given by (3).

Using the Fourier expansion (54) it is possible to determine other relevant physical quantities. For instance, the charge operator $\hat{Q} = e \int dz\, \hat{\psi}^\dagger \hat{\psi}$ [31] can be written as

$$\hat{Q} = e \sum_{n=0}^{\infty} \left(\hat{b}_n^\dagger \hat{b}_n - \hat{c}_n^\dagger \hat{c}_n \right), \tag{56}$$

where the non-observable charge of the vacuum has been removed. Likewise, the momentum operator $\hat{P} = -i \int dz\, \hat{\psi}^\dagger \nabla \hat{\psi}$ [31] can be also written in an explicit form using the expansion of the Dirac oscillator field.

The Feynman propagator for the Dirac oscillator field in the coordinate space is defined as [31, **?**]

$$i\mathcal{S}^F_{\alpha\beta}(z-z', t-t') = \langle 0 | \hat{T} \left(\hat{\psi}_\alpha(z,t) \hat{\bar{\psi}}_\beta(z',t') \right) |0\rangle . \tag{57}$$

Substituting (43) into (57) and taking into account the definition of the time-ordered operator $\hat{T}$, we obtain

$$\begin{aligned} i\mathcal{S}^F_{\alpha\beta}(z-z', t-t') = {}& \Theta(t-t') \sum_{n=0}^{\infty} u_{\alpha,n}(z) \bar{u}_{\beta,n}(z') e^{-iE_n(t-t')} \\ & - \Theta(t'-t) \sum_{n=1}^{\infty} \nu_{\alpha,n}(z) \bar{\nu}_{\beta,n}(z') e^{iE_n(t-t')}. \end{aligned} \tag{58}$$

Now we will obtain the Feynman propagator in the momenta space. To do it, we consider firstly that the Fourier transformation for the Hermite polynomials can be written as

$$\mathfrak{F}\left\{ e^{-\frac{x^2}{2}} H_n(x) \right\} = (-i)^n e^{-\frac{k^2}{2}} H_n(k), \tag{59}$$

and secondly, that the contour integral for the energy eigenvalues is written as [21]

$$i \oint_C \frac{dp_o}{2\pi} \frac{e^{-ip_o(t-t')}}{p_0^2 - p_n^2} = \Theta(t-t') \frac{e^{-iE_n(t-t')}}{2E_n} + \Theta(t'-t) \frac{e^{iE_n(t-t')}}{2E_n}, \tag{60}$$

where $p_n^2 = 2\,|n|\, m\omega + m^2$ and E_n is given by (19). Then we find that

$$\mathcal{S}^F_{\alpha\beta}(z-z', t-t') = \oint_C \frac{dp_o\, dp_z\, dp_{z'}}{(2\pi)^3} \mathcal{S}^F_{\alpha\beta}(p_o, p_z, p_{z'}) e^{-i(p_z z + p_{z'} z')}, \tag{61}$$

where the Feynman propagator in the momenta space for the one-dimensional Dirac oscillator field is given by

$$\mathcal{S}^F_{\alpha\beta}(p_o, p_z, p_{z'}) = \sum_{n=0}^{\infty} \frac{(-1)^n}{p_0^2 - p_n^2} \left[u_{\alpha,n}(p_z)\bar{u}_{\beta,n}(p_{z'}) - v_{\alpha,n-1}(p_z)\bar{v}_{\beta,n-1}(p_{z'}) \right]. \tag{62}$$

This result is in agreement with the presented in [21], where the Feynman propagator of the one-dimensional Dirac oscillator was obtained using the path integral formalism and working with a different representation of the Dirac matrices.

4. Dirac oscillator in the (3+1) dimensional case

In this section we will consider the Dirac oscillator in (3+1) dimensions. Initially we will obtain the energy spectrum and the wave functions describing the quantum states of this system. Next we develop the canonical quantization procedure for the Dirac oscillator field in the 3+1 dimensional case.

4.1 Spectrum and wave functions in (3+1) dimensions

In order to obtain the Dirac oscillator equation in the (3+1) dimensional case, we perform the following substitution which is analogous to the minimal substitution [4, 7, 14]

$$\hat{\mathbf{p}} \rightarrow \hat{\mathbf{p}} - im\omega\gamma^0\hat{\mathbf{r}}, \tag{63}$$

where $\hat{\mathbf{p}} = (\hat{p}_x, \hat{p}_y, \hat{p}_z)$ and $\hat{\mathbf{r}} = (\hat{x}, \hat{y}, \hat{z})$. The Dirac oscillator equation has the form

$$i\frac{\partial}{\partial t}\left|\psi\right\rangle = \left(\boldsymbol{\alpha}\cdot\left(\hat{\mathbf{p}} - im\omega\gamma^0\hat{\mathbf{r}}\right) + \beta m\right)\left|\psi\right\rangle. \tag{64}$$

The solutions of the Dirac oscillator equations are described by the spinor states $|\psi\rangle$, that can be written as

$$|\psi\rangle = \begin{pmatrix} |\varphi\rangle \\ |\chi\rangle \end{pmatrix}, \tag{65}$$

where $|\varphi\rangle$ and $|\chi\rangle$ are bispinors. The bispinors obey the following equations

$$(\hat{\mathbf{p}}^2 + (m\omega)^2\hat{\mathbf{r}}^2)|\varphi >= (E^2 - m^2 + (3 + 4\hat{\mathbf{s}}\cdot\hat{\mathbf{L}})m\omega)|\varphi >, \tag{66a}$$

$$(\hat{\mathbf{p}}^2 + (m\omega)^2\hat{\mathbf{r}}^2)|\chi >= (E^2 - m^2 - (3 + 4\hat{\mathbf{s}}\cdot\hat{\mathbf{L}})m\omega)|\chi > . \tag{66b}$$

From these equations, it is evident that the Dirac oscillator in the (3+1) dimensional case presents a strong spin-orbit coupling term [7, 14]. It has been demonstrated that the Dirac oscillator is a system where the angular momentum is conserved [7, 14], thus the quantum numbers of full angular momentum j and parity are good quantum numbers. Therefore, it is convenient to separate the energy eigenfunctions in two parts: radial and

angular. The spatial coordinate representation of the states (65) allows to the spinor wave functions given by

$$\psi_{n,\kappa,g}(\vec{r}) = \frac{1}{r}\begin{pmatrix} F_{n,\kappa}(r)\mathcal{Y}_{\kappa,g}(\theta,\phi) \\ iG_{n,\kappa}(r)\mathcal{Y}_{-\kappa,g}(\theta,\phi)\end{pmatrix}, \tag{67}$$

where n is the principal quantum number, κ is a quantum number related with the angular momentum and parity, and g is a quantum number related to the projection of the angular momentum in the z axis. The quantum number κ is defined as [35]

$$\kappa = \mp\left(j+\frac{1}{2}\right) = \begin{cases} -(l+1), & \text{if} \quad j = l+\frac{1}{2}, \\ l, & \text{if} \quad j = l-\frac{1}{2}. \end{cases} \tag{68}$$

The full angular momentum j takes values $j = l \pm \frac{1}{2}$, where l is the angular momentum and $\frac{1}{2}$ is the spin of the fermion. The angular momentum l' associated to the upper and lower components is [35]

$$l' = 2j - l = \begin{cases} l+1, & \text{if} \quad j = l+\frac{1}{2}, \\ l-1, & \text{if} \quad j = l-\frac{1}{2}, \end{cases} \tag{69}$$

The spinorial spherical harmonics $\mathcal{Y}_{\kappa,g}(\theta,\phi)$ and $\mathcal{Y}_{-\kappa,g}(\theta,\phi)$ are given by [35]

$$\mathcal{Y}_{\kappa,g}(\theta,\phi) = \begin{pmatrix} \sqrt{\frac{j+g}{2j}}Y_{l,g-\frac{1}{2}}(\theta,\phi) \\ \sqrt{\frac{j-g}{2j}}Y_{l,g+\frac{1}{2}}(\theta,\phi)\end{pmatrix}, \tag{70a}$$

$$\mathcal{Y}_{-\kappa,g}(\theta,\phi) = \begin{pmatrix} -\sqrt{\frac{j-g+1}{2j+2}}Y_{l',g-\frac{1}{2}}(\theta,\phi) \\ \sqrt{\frac{j+g+1}{2j+2}}Y_{l',g+\frac{1}{2}}(\theta,\phi)\end{pmatrix}. \tag{70b}$$

Substituting (67) into the Dirac equation (64) represented in coordinate space, we can find the following coupled equation system for the radial functions $F_{n,\kappa}(r)$ and $G_{n,\kappa}(r)$ [15]

$$\left[\frac{d}{dr} + \frac{\kappa + m\omega r^2}{r}\right]F_{n,\kappa}(r) = (E+m)G_{n,\kappa}(r), \tag{71a}$$

$$\left[-\frac{d}{dr} + \frac{\kappa + m\omega r^2}{r}\right]G_{n,\kappa}(r) = (E-m)F_{n,\kappa}(r). \tag{71b}$$

The solutions of this equation system are [7, 14, 15]

$$F_{n,\kappa}(r) = A[\sqrt{m\omega}r]^{l+1}e^{-\frac{m\omega r^2}{2}}L_n^{l+\frac{1}{2}}(m\omega r^2), \tag{72a}$$

$$G_{n,\kappa}(r) = \pm\mathrm{sgn}(\kappa)A'[\sqrt{m\omega}r]^{l'+1}e^{-\frac{m\omega r^2}{2}}L_n^{l'+\frac{1}{2}}(m\omega r^2), \tag{72b}$$

where $L_n^l(x)$ are the Laguerre associated polynomials and A, A' are normalization constants given by

$$A = \left[\frac{\sqrt{m\omega}\,|n|!\,(E_{n,\kappa}+m)}{\Gamma(|n|+l+3/2)E_{n,\kappa}} \right]^{\frac{1}{2}}, \tag{73a}$$

$$A' = \left[\frac{\sqrt{m\omega}\,|n'|!\,(E_{n,\kappa}-m)}{\Gamma(|n'|+l'+3/2)E_{n,\kappa}} \right]^{\frac{1}{2}}, \tag{73b}$$

where the quantum number $|n'|$ takes the following values

$$|n'| = \begin{cases} |n|-1, & \text{for} \quad \kappa < 0 \\ |n|\,, & \text{for} \quad \kappa > 0 \end{cases}. \tag{74}$$

The radial functions $F(r)$ and $G(r)$ obtained here have the same radial structure as the one associated to the non-relativistic three-dimensional harmonic oscillator. On the other hand, the energy spectrum for this case depends on the κ value. It is possible obtain that the energy eigenvalues are [15]

- For $\kappa < 0$

$$E_{n,\kappa} = \pm\sqrt{m^2 + 4\,|n|\,m\omega}, \tag{75}$$

  where the quantum number n can take the values $n = 0, \pm1, \pm2, \ldots$, and the positive sign is chosen for $n \geq 0$, meanwhile the negative sign is chosen for $n < 0$.
- For $\kappa > 0$

$$E_{n,\kappa} = \pm\sqrt{m^2 + 4\left(|n| + l + \frac{1}{2}\right) m\omega}, \tag{76}$$

  where $n = \pm0, \pm1, \pm2, \ldots$ and the sign is chosen as was mentioned in the before item. Moreover, the energy spectrum satisfies the following symmetry condition $E_{-n,\kappa} = -E_{n,\kappa}$, except for $n = 0$ which means $\kappa < 0$. For this case, it is necessary to differentiate between the quantum numbers $n = +0$ and $n = -0$ [15].

With the purpose of implementing the canonical quantization procedure for the Dirac oscillator in (3+1) dimension, we observe that the spinorial wave functions (65) satisfy the following orthonormality and completeness relations [15]

$$\sum_{\substack{\kappa=-\infty \\ \kappa\neq 0}}^{\infty} \sum_{g=-|\kappa|+\frac{1}{2}}^{|\kappa|-\frac{1}{2}} \sum_{n=-\infty}^{\infty} \psi_{n,\kappa,g}(\vec{r})\psi^{\dagger}_{n,\kappa,g}(\vec{r}\,') = \delta^3(\vec{r}-\vec{r}\,')\mathbb{I}_4, \tag{77a}$$

$$\int d^3r\,\psi^{\dagger}_{n,\kappa,g}(\vec{r})\psi_{n',\kappa',g'}(\vec{r}) = \delta_{n,n'}\delta_{\kappa,\kappa'}\delta_{g,g'}, \tag{77b}$$

respectively. Finally, we note that the covariant Dirac oscillator equation for the (3+1) dimensional case has the same form as the equation (33) for the (1+1) dimensional case.

4.2 Canonical quantization in (3+1) dimensions

To proceed with the canonical quantization for the Dirac oscillator field in the (3+1) dimensional case, we observe that the form of the Lagrangian density for this case is similar to the one in the (1+1) dimensional case and it is given by

$$\mathcal{L} = \bar{\psi}(i\gamma^{\mu}\partial_{\mu} - m)\psi + \bar{\psi}\sigma^{\mu\nu}\psi\, F_{\mu\nu}. \tag{78}$$

As in the (1+1) dimensional case, the Lagrangian density depends explicitly on the spatial coordinates and then it is not possible to obtain the energy-momentum tensor of the system. However, by means of a Legendre transformation, we can obtain the following Hamiltonian density $\mathcal{H} = \psi^{\dagger}(-i\boldsymbol{\alpha}\cdot(\vec{\nabla} + m\omega\beta\vec{r}) + \beta m)\psi$. Thus, the Hamiltonian of the field is

$$H = \int d^3r\, \psi^{\dagger}(-i\boldsymbol{\alpha}\cdot(\vec{\nabla} + m\omega\beta\vec{r}) + \beta m)\psi. \tag{79}$$

To perform the canonical quantization for this case, as is usual, the fields are considered as field operators. Then we establish the following Jordan-Wigner commutation relations

$$\left\{\hat{\psi}_{\alpha}(\vec{r},t), \hat{\psi}^{\dagger}_{\beta}(\vec{r}',t)\right\} = \delta_{\alpha\beta}\delta^3(\vec{r}-\vec{r}'), \tag{80}$$

$$\left\{\hat{\psi}_{\alpha}(\vec{r},t), \hat{\psi}_{\beta}(\vec{r}',t)\right\} = 0, \tag{81}$$

$$\left\{\hat{\psi}^{\dagger}_{\alpha}(\vec{r},t), \hat{\psi}^{\dagger}_{\beta}(\vec{r}',t)\right\} = 0, \tag{82}$$

that allow us to write the Hamiltonian operator as

$$\hat{H} = \int d^3r\, \hat{\psi}^{\dagger}(-i\boldsymbol{\alpha}\cdot(\vec{\nabla} + m\omega\beta\vec{r}) + \beta m)\hat{\psi}. \tag{83}$$

Now we can expand the field operator $\hat{\psi}(\vec{r},t)$, using the spinor wave functions (65), as follows

$$\hat{\psi}(\vec{r},t) = \sum_{\substack{\kappa=-\infty \\ \kappa\neq 0}}^{\infty} \sum_{g=-|\kappa|+\frac{1}{2}}^{|\kappa|-\frac{1}{2}} \sum_{n=-\infty}^{\infty} \hat{b}_{n,\kappa,g}\psi_{n,\kappa,g}(\vec{r})e^{-iE_{n,\kappa}t}, \tag{84}$$

where the operators $\hat{b}_{n,\kappa,g}$ and $\hat{b}^{\dagger}_{n,\kappa,g}$, respectively, annihilates and creates fermions in a state defined by the quantum numbers n, κ and g. These operators obey the following anticommutation relations

$$\left\{\hat{b}_{n,\kappa,g}, \hat{b}^{\dagger}_{n'\kappa'g'}\right\} = \delta_{n,n'}\delta_{\kappa,\kappa'}\delta_{g,g'}, \tag{85}$$

$$\left\{\hat{b}_{n,\kappa,g}, \hat{b}_{n'\kappa'g'}\right\} = 0, \tag{86}$$

$$\left\{\hat{b}^{\dagger}_{n,\kappa,g}, \hat{b}^{\dagger}_{n'\kappa'g'}\right\} = 0, \tag{87}$$

Using the properties of the energy eigenfunctions (77a) and (77b) into the Hamiltonian operator (84), we can obtain

$$\hat{H} = \sum_{\substack{\kappa=-\infty \\ \kappa\neq 0}}^{\infty} \sum_{g=-|\kappa|+\frac{1}{2}}^{|\kappa|-\frac{1}{2}} \sum_{n=-\infty}^{\infty} E_{n,\kappa} \hat{b}^{\dagger}_{n,\kappa,g} \hat{b}_{n,\kappa,g}. \tag{88}$$

This operator can be rewritten by splitting the positive and negative energy contributions and taking into account the two types of spectrum depending on the κ value. In this way, we obtain

$$\begin{aligned} \hat{H} = & \sum_{\kappa=1}^{\infty} \sum_{g} \sum_{n=0}^{\infty} E_{n,\kappa} \hat{b}^{\dagger}_{n,\kappa,g} \hat{b}_{n,\kappa,g} - \sum_{\kappa=1}^{\infty} \sum_{g} \sum_{n=0}^{\infty} E_{n,\kappa} \hat{b}^{\dagger}_{-n,\kappa,g} \hat{b}_{-n,\kappa,g} \\ & + \sum_{\kappa=1}^{\infty} \sum_{g} \sum_{n=0}^{\infty} E_{n,-\kappa} \hat{b}^{\dagger}_{n,-\kappa,g} \hat{b}_{n,-\kappa,g} - \sum_{\kappa=1}^{\infty} \sum_{g} \sum_{n=1}^{\infty} E_{n,-\kappa} \hat{b}^{\dagger}_{-n,-\kappa,g} \hat{b}_{-n,-\kappa,g}, \end{aligned} \tag{89}$$

where we have used the convention $E_{n,-\kappa} \equiv E_{n,\kappa}$, for $\kappa < 0$, and we have omitted the limits of the sum over g. It is evident that this Hamiltonian is not defined positively as happened in the (1+1) dimensional case. Again we use the picture of the Dirac's sea in order to solve this problem. To do it we take the vacuum state in an analogous way as it was defined in (49)

$$|0\rangle = \prod_{\substack{\kappa=-\infty \\ \kappa\neq 0}}^{\infty} \prod_{g} \prod_{n=0}^{\infty} \hat{b}^{\dagger}_{-n,\kappa,g} |0_D\rangle . \tag{90}$$

Therefore we can rewrite the Hamiltonian operator without considering the vacuum energy in the following way

$$\begin{aligned} \hat{H}' = & \sum_{\kappa=1}^{\infty} \sum_{g} \sum_{n=0}^{\infty} E_{n,\kappa} \hat{b}^{\dagger}_{n,\kappa,g} \hat{b}_{n,\kappa,g} + \sum_{\kappa=1}^{\infty} \sum_{g} \sum_{n=0}^{\infty} E_{n,\kappa} \hat{b}_{-n,\kappa,g} \hat{b}^{\dagger}_{-n,\kappa,g} \\ & + \sum_{\kappa=1}^{\infty} \sum_{g} \sum_{n=0}^{\infty} E_{n,-\kappa} \hat{b}^{\dagger}_{n,-\kappa,g} \hat{b}_{n,-\kappa,g} \sum_{\kappa=1}^{\infty} \sum_{g} \sum_{n=1}^{\infty} E_{n,-\kappa} \hat{b}_{-n,-\kappa,g} \hat{b}^{\dagger}_{-n,-\kappa,g}. \end{aligned} \tag{91}$$

We observe that this Hamiltonian operator describes two different types of particles and antiparticles, because particles and antiparticles have different energy spectrums depending on the sign of the κ value. This value depends explicitly on the full angular momentum j due to the value of j is based on the spin state of the fermion. In this way, we perform the following canonical transformations

$$\hat{b}^{\dagger}_{n,\kappa,g} = \hat{b}^{\dagger}_{n,\kappa,g}, \tag{92a}$$

$$\hat{b}_{-n,\kappa,g} = \hat{c}^{\dagger}_{n,\kappa,g}, \tag{92b}$$

$$\hat{b}^{\dagger}_{n,-\kappa,g} = \hat{d}^{\dagger}_{n,\kappa,g}, \tag{92c}$$

$$\hat{b}_{-n,-\kappa,g} = \hat{f}^{\dagger}_{n,\kappa,g}, \tag{92d}$$

where $\hat{b}^{\dagger}_{n,\kappa,g}$ is the creation operator of particles with full angular momentum $j = l - \frac{1}{2}$, i. e. particles with $\kappa > 0$ [32]; $\hat{c}^{\dagger}_{n,\kappa,g}$ is the creation operator of antiparticles with full angular momentum $j = l - \frac{1}{2}$, i. e. antiparticles with $\kappa > 0$ [32]; $\hat{d}^{\dagger}_{n,\kappa,g}$ is the creation operator of particles with full angular momentum $j = l + \frac{1}{2}$, i. e. particles with $\kappa < 0$ [32]; $\hat{f}^{\dagger}_{n,\kappa,g}$ is the creation operator of antiparticles with full angular momentum $j = l + \frac{1}{2}$, i. e. antiparticles with $\kappa < 0$ [32].

Finally, we can obtain that the field operator for this case can be written by means of the expansion

$$\begin{aligned}\hat{\psi}(\vec{r},t) = &\sum_{\kappa=1}^{\infty}\sum_{g}\sum_{n=0}^{\infty}\hat{b}_{n,\kappa,g}\psi_{n,\kappa,g}(\vec{r})e^{-iE_{n,\kappa}t} + \sum_{\kappa=1}^{\infty}\sum_{g}\sum_{n=0}^{\infty}\hat{c}^{\dagger}_{n,\kappa,g}\psi_{-n,\kappa,g}(\vec{r})e^{iE_{n,\kappa}t} \\ &+ \sum_{\kappa=1}^{\infty}\sum_{g}\sum_{n=0}^{\infty}\hat{d}_{n,\kappa,g}\psi_{n,-\kappa,g}(\vec{r})e^{-iE_{n,-\kappa}t} + \sum_{\kappa=1}^{\infty}\sum_{g}\sum_{n=1}^{\infty}\hat{f}^{\dagger}_{n,\kappa,g}\psi_{-n,-\kappa,g}(\vec{r})e^{iE_{n,-\kappa}t}.\end{aligned} \tag{93}$$

In an analogous way as was showed for the (1+1) dimensional case, starting from the expansion for the field operator (93), it is possible to obtain the different relevant physical quantities associated to the Dirac oscillator field in (3+1) dimensions.

5. Conclusions

In this work we have performed the canonical quantization of the Dirac oscillator field in (1+1) and (3+1) dimensions. This quantization has been possible because the solutions of the Dirac oscillator equation do not present the Klein paradox [11]. If this fact would not have satisfy, then it had not been possible to distinguish between positive and negative energy states [33], which had restricted the possibility to perform a Fourier expansion of the field operator. The Dirac oscillator field has been quantized following a similar procedure as if this field were free [31]. However, this procedure implies differences with respect to the Dirac free field quantization. For instance, the Dirac oscillator is an interacting system in which there exist bound states define with specific quantum numbers [9]. In this case the field operators create and annihilate fermions with well determined energy values in contrast to what happens in the free field case where the states have a well defined momentum [22]. Moreover, for this case, the momentum operator depends explicitly on the time thus the fermions created do not have a determined momentum. We have found that for the Dirac oscillator field it has been impossible to obtain the energy-momentum tensor due to the fact that the Lagrangian density has an explicit dependence on the spatial coordinates. Nevertheless, this problem could be solved by the introduction in the system of an additional field which describes the interaction. We have also obtained the Feynman propagator in the (1+1) dimensional case in agreement with the result obtained in the literature by using an functional procedure [21]. We note that for the Dirac oscillator field in the (3+1) dimensional case, we have found that there exist two types of particles and antiparticles because there are two possible values for the full

angular momentum. Finally, we have found that while for the free Dirac field the energy quanta of the infinite harmonic oscillators are relativistic energies of free particles, for the Dirac oscillator quantum field the quanta are energies of relativistic linear harmonic oscillators. The canonical quantization procedure for the Dirac oscillator field in (2+1) dimensions is an exercise to develop explicitly. We consider that the possibility to study the Dirac oscillator as a quantum field opens the doors to future applications in different areas of the Physics.

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Unitary Evolution and Uniqueness of the Fock Quantization in Flat Cosmologies with Compact Spatial Sections

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Abstract: We study the Fock quantization of scalar fields with a time dependent mass in cosmological scenarios with flat compact spatial sections. This framework describes physically interesting situations like, e.g., cosmological perturbations in flat Friedmann-Robertson-Walker spacetimes, generally including a suitable scaling of them by a background function. We prove that the requirements of vacuum invariance under the spatial isometries and of a unitary quantum dynamics select (a) a unique canonical pair of field variables among all those related by time dependent canonical transformations which scale the field configurations, and (b) a unique Fock representation for the canonical commutation relations of this pair of variables. Though the proof is generalizable to other compact spatial topologies in three or less dimensions, we focus on the case of the three-torus owing to its relevance in cosmology, paying a especial attention to the role played by the spatial isometries in the determination of the representation.

Keywords: Fock Quantization; Flat Cosmologies; Flat Friedmann-Robertson-Walker spacetimes; Filed Theory
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1. Introduction

The construction of a quantum theory to describe a classical system is a process plagued with many ambiguities. Generically, the correspondence is not one-to-one. What is more important, the quantum physics depends on the choices made in the steps affected by those ambiguities. The specification of a unique quantization must then either rest on the confrontation of the predictions with experiments, or be achieved by appealing to other kinds of criteria, usually related to symmetries of the system or to the behavior of the quantum states. The problem is especially relevant in cosmology, both because the windows for quantum effects in cosmological observations are certainly narrow (if any), and because one cannot really select the best candidate for a quantum model of the universe by performing an indefinite number of repeated measurements (in copies of the same state), since we can only observe the universe in which we live.

Even if one identifies a classical system with a specific algebra of observables, attained by first selecting a set of appropriate variables for the system and constructing out of them the algebra of functions which we declare of interest, its quantization is still not unique. In general, there will exist representations of that algebra which are not equivalent (that is, which are not related by a unitary transformation). This problem appears not just for generic systems with complicated phase spaces or interactions, but also for the simplest systems with a linear behavior. In the case of standard quantum mechanics, when the number of degrees of freedom is finite, the ambiguities in the representation can be removed by imposing additional requirements. For the Weyl algebra corresponding to the free particle, one usually demands that the representation be irreducible, unitary and strongly continuous. Then, the Stone-von Neumann theorem guarantees that the representation is unique (up to unitary transformations) [1]. A similar result does not exist when one considers systems with fields, which have an infinite number of degrees of freedom. The ambiguities persist even if one takes advantage of the linear structures and restricts the consideration to representations of the Fock type, where a concept of particle and vacuum are available (either with a genuine physical interpretation or just as mathematical entities). The possible choices of Fock representation which are physically different are still infinite, and correspond to non-equivalent choices of a vacuum [2].

For fields propagating in highly symmetric spacetimes, the symmetries of the background can be employed as criteria to determine the quantization [2, 3, 4], asking e.g. that all the quantum structures incorporate those symmetries. For instance, this happens in Minkowski spacetime, where the vacuum can be selected by demanding Poincaré invariance [2]. But, for more general spacetimes, no generic criterion exists that specifies

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the Fock representation. In generic situations, there is not enough spacetime symmetry to fix it. Therefore, supplementary or alternate requirements are needed in order to pick out a unique equivalence class of Fock representations. Frequently, the situation found in cosmology is that the spatial sections still present a high degree of symmetry (at least in a certain approximation), but the stationarity is lost owing to the universe expansion (or contraction). In this framework, it seems natural to adhere to the remaining spatial symmetries, demanding that they are naturally included in the quantum theory, and replace the criterion of time symmetry with the closest possible one, namely, with the requirement of a unitary evolution. Unitarity will guarantee a standard probabilistic interpretation in the quantum theory, though the loss of time symmetry will make the vacuum change dynamically. Actually, the combined criterion of invariance under the spatial symmetries and a unitary evolution has been successfully employed recently in the selection of a unique quantization for cosmological models.

Apart from the choice of representation, there is an additional ambiguity that arises naturally in the construction of a Fock description for fields in cosmological scenarios. The non-stationarity of the spacetime leads to the obvious possibility of absorbing part of the time dependence of the field via its scaling by a function that depends only on the background. This is so irrespective of whether the spacetime in which the propagation takes place is a true physical background [5, 6], an effective background (emerging from an effective description of the system, for instance incorporating some quantum modifications at an effective level [7, 8, 9, 10]) or an auxiliary background (like, e.g., in dimensional reductions of gravitational systems in General Relativity, using the presence of Killing symmetries [11, 12, 13]). Scalings of this type are found in many circumstances when considering fields or cosmological perturbations around Friedmann-Robertson-Walker (FRW) spacetimes, for instance, like in the case of test fields, or scalar and tensor perturbations, including their description in terms of gauge invariants, as with the Mukhanov-Sasaki variables [14]. These scalings render the field equations in the form of those corresponding to a Klein-Gordon (KG) field in a static (auxiliary) spacetime. The time dependence still shows up via the appearance of a varying mass term. Besides, usually there is no friction term in the scaled equation (or one can neglect it when considering short scales in the field behavior). In particular, it was demonstrated recently that, for KG equations in conformally ultrastatic spacetimes, one can always find an adequate scaling which, properly combined with a change of time, removes the term proportional to the first time derivative in the field equations [15]. The scaling of the field configuration can always be regarded as part of a time dependent canonical transformation in the system, which is natural to consider as a local and linear transformation in order not to spoil this type of properties in the field system. In such a transformation, the field momentum gets the inverse scaling of that of the field configuration [to maintain the canonical commutation relations (CCR's)] and, in addition, may admit an extra contribution linear in the field configuration, multiplied by a function of time. Any of these canonical transformations leads to a new canonical pair for the field, but also alters the dynamics, since the change is time dependent. The quantum description can be made by adopting any of these

canonical pairs for the field, introducing a new kind of ambiguity with infinitely many possibilities.

As we have briefly commented, the criterion of invariance under the spatial symmetries of the classical spacetime, together with the unitary implementation of the dynamics in the quantum realm, has been employed in cosmological systems to remove the ambiguity in the Fock quantization. Indeed, this criterion has shown useful not only to select a Fock representation, but also to determine a unique canonical pair for the field among all those that can be reached by means of linear canonical transformations which depend on time. Note that, since these transformations modify the dynamics, the fact that the evolution is unitary is intimately related to the canonical pair adopted. The criterion was first applied to the case of the Gowdy models [12, 13, 16, 17, 18, 19, 20]: reductions of General Relativity with two spatial Killing vectors and compact spatial sections. They describe gravitational waves propagating in cosmological spacetimes with compact universes. In the case of linearly polarized waves, the system admits a description in terms of a KG field (with time dependent mass) in an auxiliary, dimensionally reduced stationary spacetime. The spatial sections of this auxiliary background can be either isomorphic to the circle, S^1, or to the sphere, S^2, depending on the spatial topology of the Gowdy model (which can be that of a three-torus, of a three-sphere, or of a three-handle, $S^2 \times S^1$). The conclusions about the validity of the proposed criterion for the choice of a unique Fock quantization in the Gowdy models have been lately extended to the case of backgrounds with spatial sections isomorphic to d-dimensional spheres, with $d \leq 3$ [21, 22, 23, 24], and even more recently to a general compact topology in three or less spatial dimensions [25, 26]. Although this generalization proves that the uniqueness holds in any case with compact spatial sections, we will focus here on the case when these sections have the topology of a three-torus. The relevance of this case is clear since it describes flat universes, which is precisely the favored scenario for the universe according to observations. Moreover, the discussion for fields in spacetimes with generic compact topology [25, 26] is obscured by the fact that there may be no clear geometric interpretation of the considered symmetry group, in the most general case. This interpretation is neat in the case of the three-torus. Besides, there exist certain peculiarities which motivate the mathematical interest in the analysis of the three-torus. Namely, since the group of isometries of the three-torus is an Abelian compact group (as in the S^1 case [22]), its irreducible (unitary) representations are one-dimensional and defined over complex vector spaces [27]. This introduces certain subtleties in the characterization of the Fock representations which are compatible with those symmetries. These subtleties arise because the complex representations of the symmetry group must be combined suitably so that, at the end of the day, one deals exclusively with real scalar fields. In order to fully take into account this issue, and show explicitly how those complex representations combine, a departure from the general approach followed for other topologies in the literature is adopted [21, 23]. Explaining the characterization of the symmetric representations will be one of the main goals of this work.

The paper is organized as follows. In Sec. 2. we start by considering a KG field

with time dependent mass in a static spacetime with flat spatial sections of three-torus topology. We briefly present the classical system and the standard procedure to introduce a Fock quantization of its associated phase space. Then, in Sec. 3. we prove that our combined criterion of a) invariance under the isometries of the three-torus and b) unitary implementation of the dynamics, selects a unique Fock representation, up to unitary transformations. This unitary class of representations includes the one which would be naturally adopted if the field had vanishing mass. Our proof contains a detailed discussion of the characterization of the representations which are invariant under the symmetries of the three-torus, with a careful treatment of the symmetry transformations and the consistency conditions coming from the reality of the field. We go beyond this result of uniqueness in Sec. 4., where we analyze time dependent canonical transformations arising from a scaling of the field configuration, and demonstrate that only one of all those transformations is compatible with our requirements of spatial symmetry invariance and unitary evolution: the trivial one. Finally, we present our conclusions in Sec. 5.. The discussion in this work follows lines of arguments similar to those presented in Refs. [26, 28], to which we refer the reader for further details about the analysis.

2. Klein-Gordon field with time dependent mass

2.1 The classical model

Before proving that the criterion that we put forward indeed selects a unique class of unitarily equivalent Fock representations for a linear scalar field with time dependent mass in a flat spacetime with compact sections, let us start by describing the classical set-up of our theory.

We consider a real scalar field φ defined on a flat spacetime whose spatial sections have the topology of a three-torus, T^3. These sections are equipped with the standard spatial metric of the three-torus, $h_{ij}(i,j=1,2,3)$. The field is subject to a linear equation of KG type:

$$\ddot{\varphi} - \Delta\varphi + s(t)\varphi = 0, \tag{1}$$

where the dot stands for the time derivative, Δ is the Laplace-Beltrami (LB) operator associated with the three-torus metric h_{ij}, and $s(t)$ can be interpreted as a time dependent mass. In principle, $s(t)$ can be any time function, and only later in the discussion we will impose on it some extremely mild conditions about its derivatives. On the other hand, the time domain in which the KG field is defined can be any arbitrary connected real interval $\mathbb{I} \subset \mathbb{R}$. We do not impose that $\mathbb{I}$ be the real line, not even that it be unbounded. No specific form for the interval $\mathbb{I}$ is assumed. This is important for the applications of our results to situations in which the field description is effective, since the validity of the effective spacetime geometry can be restricted just to a certain time interval. More generally, the domain $\mathbb{I}$ might even be just a union of connected components. In such cases, the restriction to one of the components is sufficient to achieve the same uniqueness results.

This equation can be obtained in very different cosmological models. An important class of systems for which it has a major relevance are scalar fields propagating in non-stationary, cosmological spacetimes, like e.g. the case of matter fields in inflationary backgrounds [5, 29, 30]. Another class are cosmological perturbations around FRW spacetimes. According to our comments in Sec. 1., a suitable choice of the time parametrization and a time-dependent scaling leads the field equations of those systems to a KG equation of the above type [23, 24]. In this cosmological context, the case of flat spatial sections that we study here is the most interesting one owing to its potential applications to describe situations in the observed universe.

Clearly, this dynamical equation is invariant under the group of isometries of the three-torus, since the LB operator is defined in terms of the standard metric on T^3. To incorporate and analyze the role of these isometries, we consider the composition of rigid rotations in each of the periodic spatial directions θ_i that diagonalize the metric h_{ij},

$$T_{\alpha_i} : \theta_i \to \theta_i + \alpha_i, \quad \forall \alpha_i \in S^1. \tag{2}$$

Here, α_i is the angle parameter that provides the rotation in the direction i ($i = 1, 2, 3$). We will call $T_{\vec{\alpha}}$, with $\vec{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$, the transformation obtained by composing the corresponding rotations.

The canonical phase space Γ of the field system is obtained from the Cauchy data at a reference time $t_0 \in \mathbb{I}$, namely $\{(\varphi, P_\varphi)\} = \{(\varphi_{|t_0}, \dot{\varphi}_{|t_0})\}$, equipped with a symplectic structure Ω that amounts to the canonical Poisson brackets $\{\varphi(\vec{\theta}), P_\varphi(\vec{\theta'})\} = \delta^3(\vec{\theta} - \vec{\theta'})$, with $\delta^3(\vec{\theta})$ being the Dirac delta on T^3 and $\vec{\theta}$ the spatial point with coordinates θ_i. Note that we assume that the Hamilton equation for the field momentum is $P_\varphi = \dot{\varphi}$.[2]

As a result of the periodicity on the spatial coordinates θ_i, one can decompose the field φ (and its momentum) in an expansion in Fourier modes. Note that these modes are eigenfunctions of the LB operator. The field decomposition using these complex eigenfunctions takes the form

$$\varphi(t, \vec{\theta}) = \frac{1}{(2\pi)^{3/2}} \sum_{\vec{m}} \mathfrak{q}_{\vec{m}}(t) \exp\{i(\vec{m} \cdot \vec{\theta})\}, \tag{3}$$

where $\vec{m}$ is the tuple of integers (m_1, m_2, m_3) (i.e., $m_i \in \mathbb{Z}$, for $i = 1, 2, 3$), and we have introduced the notation $\vec{m} \cdot \vec{\theta} = \sum_i m_i \theta_i$. Recalling that the field is real, we find that the coefficients of the expansion are subject to the reality conditions

$$\mathfrak{q}_{-\vec{m}}(t) = [\mathfrak{q}_{\vec{m}}(t)]^*, \tag{4}$$

where the symbol $*$ denotes complex conjugation.

This complex decomposition is well adapted to the three-torus symmetries. The Fourier modes are eigenfunctions of all the transformations $T_{\vec{\alpha}}$. Moreover, each tuple $\vec{m}$ provides a different, inequivalent irreducible representation of the isometry group. In

[2] There is no problem with the density weight of the field momentum in this equation, since the standard metric of the three-torus has a unit determinant.

particular, we straightforwardly see that such irreducible complex representations are one-dimensional, as it corresponds to the case under consideration, with symmetries that form an Abelian compact group. The disadvantage of using this complex Fourier decomposition, nonetheless, is that we have to deal with the complications posed by the reality conditions. In order to avoid these complications, and clarify how the irreducible representations over complex vector spaces combine in the case of real fields, it is then advisable to adopt an alternate decomposition in terms of real eigenfunctions of the LB operator, namely, in terms of Fourier modes corresponding to sines and cosines:

$$\varphi(t,\vec{\theta}) = \frac{1}{(\pi)^{3/2}} {\sum_{\vec{n}}}' [q_{\vec{n}}(t)\cos(\vec{n}\cdot\vec{\theta}) + x_{\vec{n}}(t)\sin(\vec{n}\cdot\vec{\theta})]. \tag{5}$$

Here, we have defined the tuple $\vec{n} = (n_1, n_2, n_3)$, with $n_i \in \mathbb{Z}$ $(i = 1, 2, 3)$. Note that we have changed the notation for the labels of the Fourier modes and coefficients from $\vec{m}$ to $\vec{n}$, to facilitate the distinction between the complex and the real formulations. The sum in the above expansion contains only tuples $\vec{n}$ of integers whose first non-zero component is positive. All different tuples which satisfy this restriction are to be summed over (and just once each). We indicate the restriction in the sum with a tilde, instead of making it explicit, something which would complicate the notation in excess. Furthermore, here and in the rest of our discussion we will ignore the zero mode, $\vec{n} = (0, 0, 0)$. As we can anticipate, the unitary evolution and the uniqueness of the representation do not depend on the removal of a finite number of degrees of freedom. This mode can always be quantized separately, including the possibility of employing non-standard methods in its quantum mechanical description (like, e.g., using loop quantization methods). Its exclusion does not alter the field properties of the system, on which we concentrate our attention.

Since we have expanded the field in eigenfunctions of the LB operator, the different modes are dynamically decoupled. For the cosine modes, we obtain

$$\ddot{q}_{\vec{n}} + [\omega_n^2 + s(t)]q_{\vec{n}} = 0, \tag{6}$$

and similarly for the sine modes, $x_{\vec{n}}$. Thus, the dynamical equations only depend on the corresponding LB eigenvalue of the mode, $-\omega_n^2$, which is given by $\omega_n^2 = \sum_i n_i^2$. The subindex n, introduced as a label in ω_n, is taken as a positive integer which designates the order of these eigenvalues. That is, $\omega_n < \omega_{n'}$ if $n < n'$. Note that, given the compactness of the three-torus, the LB eigenvalues indeed form a sequence $\{\omega_n\}$, which is certainly unbounded. It is also worth noticing that the degeneracy g_n of each eigenspace of the LB operator presents a complicated dependence on the label n, because of accidental degeneracy: one can find different tuples $\vec{n}$ (others than those related by a flip of sign in one of the components, or by permutations of the components) which lead to the same eigenvalue. For the sake of an example: the tuples $(2, 2, 1)$ and $(3, 0, 0)$ correspond to the same eigenvalue $\omega^2 = 9$.

Finally, we decompose the field momentum P_φ in the same way as we have explained for the field configuration. Then, the momentum coefficients for the sine and cosine

contributions in the real modes expansion, which we will call $p_{\vec{n}}$ and $y_{\vec{n}}$, respectively, satisfy the dynamical equations $p_{\vec{n}} = \dot{q}_{\vec{n}}$, and $y_{\vec{n}} = \dot{x}_{\vec{n}}$. The non-vanishing Poisson brackets in terms of these coefficients are $\{q_{\vec{n}}, p_{\vec{n}'}\} = \{x_{\vec{n}}, y_{\vec{n}'}\} = \delta_{\vec{n}\vec{n}'}$.

2.2 Fock quantization

The Hilbert space of the system over which we construct the quantum theory is the direct sum of the symmetric tensor products of the one-particle Hilbert space $\mathcal{H}_0$. The Fock space, then, is determined by the one-particle space. For the construction of the latter, the only ingredient needed is a complex structure J [2]. We recall that a complex structure is a real map on phase space whose square is minus the identity and that preserves the symplectic structure Ω [2]. In addition, we demand the complex and the symplectic structures to be compatible in the sense that their composition, $\Omega(J\cdot,\cdot)$ must be a positive bilinear form. The sector of positive frequency (complex) solutions is obtained with the projector $\frac{1}{2}(\mathbf{1} - iJ)$, where $\mathbf{1}$ is the identity. This sector is then completed into the one-particle Hilbert space using the norm provided by $\frac{1}{2}[\Omega(J\cdot,\cdot) - i\Omega(\cdot,\cdot)]$. Therefore, the complex structure contains all the information that is physically relevant to determine the different possible Fock representations. This can be rephrased by saying that a complex structure, together with the symplectic structure, defines a state, which is usually called the vacuum, and then a representation of the CCR's.

In order to discuss the Fock quantization of the scalar field, we now introduce a particular, well-known complex structure which we will employ as a starting point in our analysis. The chosen complex structure, J_0, is the one which would be naturally related to a free, massless KG field. Hence, it is determined entirely by the three-torus metric, via the corresponding LB operator. In particular, this property guarantees that the complex structure J_0, and thus the vacuum selected by it, is invariant under the isometry group of the three-torus. In terms of the LB operator, we can define J_0 using a decomposition in eigenmodes, introducing the annihilationlike variables

$$a_{\vec{n}} = \frac{1}{\sqrt{2\omega_n}}(\omega_n q_{\vec{n}} + i p_{\vec{n}}), \qquad \tilde{a}_{\vec{n}} = \frac{1}{\sqrt{2\omega_n}}(\omega_n x_{\vec{n}} + i y_{\vec{n}}). \tag{7}$$

The corresponding creationlike variables are given by the complex conjugates $a^*_{\vec{n}}$ and $\tilde{a}^*_{\vec{n}}$. These new variables provide a complete set of coordinates on phase space. The action of J_0 on this new basis is defined to be diagonal, in the standard form

$$J_0(a_{\vec{n}}) = i a_{\vec{n}}, \qquad J_0(\tilde{a}_{\vec{n}}) = i\tilde{a}_{\vec{n}}, \tag{8}$$

$$J_0(a^*_{\vec{n}}) = -i a^*_{\vec{n}}, \qquad J_0(\tilde{a}^*_{\vec{n}}) = -i\tilde{a}^*_{\vec{n}}. \tag{9}$$

The evolution of these variables from a fixed initial time t_0 to any another time $t \in \mathbb{I}$ can be expressed as a linear transformation U (since the field equations are linear). Given that different modes decouple in the dynamics, the evolution is actually block diagonal, with 2×2 blocks $U_{\vec{n}}$, one for each pair of annihilation and creationlike variables. Furthermore, since the dynamical equations of the modes depend only on the eigenspace of

the LB operator under consideration (labeled by n), the same happens with the mentioned blocks, which we will therefore designate by U_n. Therefore, it is straightforward to conclude that the evolution can be described in the form

$$\begin{pmatrix} a_{\vec{n}}(t) \\ a^*_{\vec{n}}(t) \end{pmatrix} = U_n \begin{pmatrix} a_{\vec{n}}(t_0) \\ a^*_{\vec{n}}(t_0) \end{pmatrix}, \quad U_n = \begin{pmatrix} \alpha_n(t,t_0) & \beta_n(t,t_0) \\ \beta^*_n(t,t_0) & \alpha^*_n(t,t_0) \end{pmatrix}, \tag{10}$$

and similarly for $(\tilde{a}_{\vec{n}}, \tilde{a}^*_{\vec{n}})$. Finally, since the dynamical evolution is a symplectic transformation, the alpha and beta functions appearing in this matrix expression must satisfy the relation

$$|\alpha_n(t,t_0)|^2 - |\beta_n(t,t_0)|^2 = 1, \quad \forall t \in \mathbb{I}, \tag{11}$$

for all values of the eigenvalue label n.

In the next section we will show that the Fock representation determined by J_0 admits a unitary implementation of the evolution and, furthermore, that this is the only representation with that property (up to unitary equivalence) among all those that are invariant under the isometry group of the three-torus. We recall that, in general, a linear canonical transformation U can be implemented quantum mechanically as a unitary transformation in the representation determined by a complex structure J if and only if the antilinear part of the transformation, given by $U_J = \frac{1}{2}(U + JUJ)$, is a Hilbert-Schmidt operator [31], namely, that the trace of $U_J^\dagger U_J$ is finite (here, the dagger denotes the adjoint operator). It is possible to rephrase this condition as the requirement that the antilinear coefficients of the considered transformation (usually called the beta Bogoliubov coefficients) be square summable (that is, that their squared norms have a convergent sum). Another equivalent way of stating this condition is to demand that the image of the vacuum state under the transformation U possess a finite number of "particles", using the particle concept associated to the original vacuum. We will use this condition for unitary implementability (in any of its versions) in the rest of our discussion.

3. Uniqueness of the quantization

We will now show that the Fock representation determined by the complex structure J_0 leads to a unitary quantum evolution, even if the field has in fact a time dependent mass. We will also characterize the most general complex structure that is invariant under the symmetry group formed by the transformations $T_{\vec{\alpha}}$. We will see that they are all related by a specific family of symplectic transformations. Using that characterization, we will prove the uniqueness of the invariant complex structure (up to unitary equivalence) under the requirement that the dynamics admit a unitary implementation. To avoid repeating parts of the demonstration which follow the line of arguments presented in the literature for the three-sphere [21], we will review the main steps of the proof and concentrate our attention just on the aspects that are specific of the three-torus.

3.1 Unitary evolution in the massless representation

According to our comments above, the evolution U is implementable as a unitary transformation on the Fock space determined by J_0 if and only if its beta coefficients $\beta_n(t, t_0)$ [appearing in Eq. (10)] are square summable. Taking into account the degeneracy g_n of the eigenspaces of the LB operator, the necessary and sufficient condition is the finiteness of the sum $\sum_n g_n|\beta_n(t, t_0)|^2$ for all possible values of time $t \in \mathbb{I}$. Physically, this amounts to a finite particle production during evolution. Clearly, the summability of the sequence of beta functions (in squared norm) depends on the asymptotic behavior of β_n (and g_n) in the ultraviolet, namely, in the limit of infinitely large eigenvalues ω_n. The asymptotic analysis of the Bogoliubov coefficients for a KG field with time dependent mass in a stationary spacetime was carried out in Ref. [21], and the result applies in particular to the case discussed here, when the compact spatial sections are isomorphic to a three-torus. The analysis leads to the conclusion that

$$\alpha_n(t, t_0) = e^{-i\omega_n(t-t_0)} + O\left(\frac{1}{\omega_n}\right), \qquad \beta_n(t, t_0) = O\left(\frac{1}{\omega_n^2}\right). \tag{12}$$

The symbol O indicates the asymptotic order. The only hypothesis about the field that is employed to deduce this behavior (but that it is not even necessary for its validity) is that the mass function $s(t)$ possesses a first derivative which is integrable in every compact subinterval of $\mathbb{I}$. Using the asymptotics, it is straightforward to see that the condition for a unitary implementation of the evolution is equivalent to the summability of the sequence formed by g_n/ω_n^4. To check whether this summability holds, one has to study how the degeneracy g_n changes with n in the limit when this label gets infinitely large. This variation of g_n is quite involved owing to the accidental degeneracy that we have already pointed out. The exact dependence of the degeneracy with n cannot be given explicitly. Nonetheless, for our discussion, we only need to compute the asymptotic behavior of g_n, a task which can be done in a relatively simple way as follows.

The values ω_n^2 can be understood as the norm of the vector $\vec{n}$ provided by the tuple that labels the modes. In principle, $\vec{n}$ is restricted so that its first non-vanishing (integer) component be positive. However, there exist two modes for each value of $\vec{n}$: the sine and cosine modes. We can assign the two modes to the couple of vectors $(\vec{n}, -\vec{n})$. It is then clear that we can make correspond modes to all vectors with integer components (except the zero-mode, that has been excluded). Let us call D_N the number of modes whose eigenvalue ω_n is in the interval $(N, N+1]$, with N a natural number. Since $1/\omega_n$ is strictly decreasing with n, we have that the sum $\sum_n (g_n/\omega_n^4)$, whose convergence we want to check, is always equal or smaller than $\sum_N (D_N/N^4)$. Geometrically, the value of D_N is the number of vertices of the cubic lattice with step equal to one that are contained between the sphere of radius $N+1$ (including its surface) and the sphere of radius N. Therefore, D_N increases with N like N^2. It is then straightforward to see that the sum of D_N/N^4 is finite, and a fortiori that of g_n/ω_n^4. Thus, the Fock representation determined by J_0, naturally associated with the case of a massless field, provides in fact a unitary implementation of the dynamics even in the presence of a time dependent mass.

3.2 Characterization of the invariant complex structures

We will now characterize the most general complex structure that is invariant under the group of symmetries formed by the transformations $T_{\vec{\alpha}}$, corresponding to the isometries of the three-torus. For simplicity, we will just call *invariant* such complex structures. To reach this characterization, we will follow a procedure which differs from that presented in the literature for other compact spatial topologies, adopting our analysis to the peculiarities of the three-torus. As we have remarked, in our case, the isometry group is Abelian, and hence its irreducible representations are all one-dimensional over complex vector spaces. On the other hand, since the scalar field studied is real, these representations must be combined suitably. To clarify this interplay, we will rather consider real representations from the start, adapted to the decomposition of the field in sine and cosine modes.

It is also worth noticing that, owing to the commented accidental degeneracy, the eigenspaces of the LB operator do not provide irreducible representations of the isometry group, even if the operator commutes with the isometries because it is constructed out of the three-torus metric. Again, this situation is novel compared to that found in the literature, e.g. for d-spheres [21, 22, 23, 24]. Hence, the case of the three-torus calls for a detailed analysis, that we present in the rest of this subsection.

The action of the three-torus symmetries $T_{\vec{\alpha}}$ on the real sine and cosine modes [see Eq. (5)] is easily derived from its active transformation of the field. One gets

$$\begin{pmatrix} q'_{\vec{n}} \\ x'_{\vec{n}} \end{pmatrix} = \begin{pmatrix} \cos(\vec{n}\cdot\vec{\alpha}) & -\sin(\vec{n}\cdot\vec{\alpha}) \\ \sin(\vec{n}\cdot\vec{\alpha}) & \cos(\vec{n}\cdot\vec{\alpha}) \end{pmatrix} \begin{pmatrix} q_{\vec{n}} \\ x_{\vec{n}} \end{pmatrix}. \tag{13}$$

Similar equations are obtained for the modes of the field momentum $(p_{\vec{n}}, y_{\vec{n}})$. We notice that these transformations only mix modes in pairs, just the sine and cosine modes with the same label $\vec{n}$. Furthermore, the action on each of these pairs is different. The sine and cosine modes that get mixed belong to the same LB eigenspace and hence have the same dynamics. Besides, as we anticipated, not all modes in the same eigenspace get mixed under the action of the symmetry group, owing to the accidental degeneracy. This fact complicates the characterization of the invariant complex structures and leads to a situation which is similar to that encountered for the S^1 topology [16, 22], though in higher dimensions. Examining the action of the transformations $T_{\vec{\alpha}}$ given above, it is not difficult to realize that the most general invariant complex structure must be block diagonal in the label $\vec{n}$, namely,

$$J = {\bigoplus_{\vec{n}}}' J_{\vec{n}}, \tag{14}$$

where each complex structure $J_{\vec{n}}$ corresponds to a 4×4 block, associated with the sine and cosine modes determined by $\vec{n}$ for the field configuration and momentum. In the direct sum, we have used the same kind of notation introduced in Eq. (5).

Let us express the blocks $J_{\vec{n}}$ in terms of smaller 2×2 blocks in the phase (sub)space

basis formed by $(q_{\vec{n}}, x_{\vec{n}}, p_{\vec{n}}, y_{\vec{n}})$:

$$J_{\vec{n}} = \begin{pmatrix} A_{\vec{n}} & B_{\vec{n}} \\ C_{\vec{n}} & D_{\vec{n}} \end{pmatrix}. \tag{15}$$

From the condition that J be an invariant complex structure, so that $T_{\vec{\alpha}}^{-1} J T_{\vec{\alpha}} = J$ for all transformations $T_{\vec{\alpha}}$, one concludes after a straightforward computation that every 2×2 block $\{Q_{\vec{n}}\} = \{A_{\vec{n}}, B_{\vec{n}}, C_{\vec{n}}, D_{\vec{n}}\}$ must commute with all the rotation matrices of the form

$$R_{\vec{n}}(\vec{\alpha}) = \begin{pmatrix} \cos(\vec{n}\cdot\vec{\alpha}) & -\sin(\vec{n}\cdot\vec{\alpha}) \\ \sin(\vec{n}\cdot\vec{\alpha}) & \cos(\vec{n}\cdot\vec{\alpha}) \end{pmatrix}, \tag{16}$$

that is

$$R_{\vec{n}}^{-1}(\vec{\alpha}) Q_{\vec{n}} R_{\vec{n}}(\vec{\alpha}) = Q_{\vec{n}}. \tag{17}$$

It then follows that every 2×2 block must have a diagonal part proportional to the identity and a skew-symmetric non-diagonal part, namely,

$$Q_{\vec{n}} = \begin{pmatrix} Q_{\vec{n}}^{(1)} & Q_{\vec{n}}^{(2)} \\ -Q_{\vec{n}}^{(2)} & Q_{\vec{n}}^{(1)} \end{pmatrix}, \tag{18}$$

where $Q_{\vec{n}}^{(1)}$ and $Q_{\vec{n}}^{(2)}$ are arbitrary real numbers.

We still have to impose the condition that the invariant complex structure be compatible with the symplectic structure, so that their combination $\Omega(J\cdot,\cdot)$ provides a positive definite bilinear map on phase space. In terms of the blocks $J_{\vec{n}}$, this condition implies that $J_{\vec{n}}^T \Omega_{\vec{n}}$ must be a positive definite symmetric matrix, where $J_{\vec{n}}^T$ is the transpose of $J_{\vec{n}}$ and the blocks of the symplectic structure are

$$\Omega_{\vec{n}} = \begin{pmatrix} \mathbf{0}_{2\times 2} & -\mathbf{1}_{2\times 2} \\ \mathbf{1}_{2\times 2} & \mathbf{0}_{2\times 2} \end{pmatrix}. \tag{19}$$

Here, $\mathbf{0}_{2\times 2}$ is the zero matrix in two dimensions and $\mathbf{1}_{2\times 2}$ is the identity matrix. In order to satisfy this requirement, the 2×2 blocks of $J_{\vec{n}}$ given by $B_{\vec{n}}$ and $C_{\vec{n}}$ must be symmetric matrices of negative and positive definite type, respectively, whereas the two other blocks must satisfy that $A_{\vec{n}} = -D_{\vec{n}}$. Together with the condition of invariance discussed above, we then conclude that the blocks $J_{\vec{n}}$ must have the form

$$J_{\vec{n}} = \begin{pmatrix} A_{\vec{n}} & B_{\vec{n}} \\ C_{\vec{n}} & -A_{\vec{n}} \end{pmatrix}, \tag{20}$$

where $B_{\vec{n}}$ and $C_{\vec{n}}$ must be proportional to the identity, with a non-positive and a non-negative constant of proportionality, respectively. That is,

$$B_{\vec{n}}^{(2)} = C_{\vec{n}}^{(2)} = 0, \qquad B_{\vec{n}}^{(1)} \leq 0, \qquad C_{\vec{n}}^{(1)} \geq 0. \tag{21}$$

In addition, we must also impose the remaining conditions that the square of the complex structure be minus the identity, $J^2 = -\mathbf{1}$, and that it leave invariant the symplectic structure, namely $\Omega(J\cdot, J\cdot) = \Omega(\cdot,\cdot)$. These requirements lead in turn to the following conditions on the blocks $J_{\vec{n}}$:

$$J_{\vec{n}}^2 = -\mathbf{1}_{4\times 4}, \qquad J_{\vec{n}}^T \Omega_{\vec{n}} J_{\vec{n}} = \Omega_{\vec{n}}. \tag{22}$$

From the former of these restrictions we obtain the following equations for the matrix elements:

$$-[A_{\vec{n}}^{(1)}]^2 + [A_{\vec{n}}^{(2)}]^2 - B_{\vec{n}}^{(1)} C_{\vec{n}}^{(1)} = 1, \qquad A_{\vec{n}}^{(1)} A_{\vec{n}}^{(2)} = 0. \tag{23}$$

On the other hand, the second restriction leads to the equations

$$[A_{\vec{n}}^{(1)}]^2 + [A_{\vec{n}}^{(2)}]^2 + B_{\vec{n}}^{(1)} C_{\vec{n}}^{(1)} = -1, \tag{24}$$

$$A_{\vec{n}}^{(2)} B_{\vec{n}}^{(1)} = 0, \qquad A_{\vec{n}}^{(2)} C_{\vec{n}}^{(1)} = 0. \tag{25}$$

Summing the first equality of Eq. (23) and Eq. (24), we get that the matrix element $A_{\vec{n}}^{(2)}$ mush vanish. Therefore, every block $J_{\vec{n}}$ of an invariant complex structure (compatible with the symplectic form) must have the form

$$J_{\vec{n}} = \begin{pmatrix} A_{\vec{n}}^{(1)} \mathbf{1}_{2\times 2} & B_{\vec{n}}^{(1)} \mathbf{1}_{2\times 2} \\ C_{\vec{n}}^{(1)} \mathbf{1}_{2\times 2} & -A_{\vec{n}}^{(1)} \mathbf{1}_{2\times 2} \end{pmatrix}, \tag{26}$$

where

$$B_{\vec{n}}^{(1)} < 0, \qquad C_{\vec{n}}^{(1)} > 0, \qquad A_{\vec{n}}^{(1)} = \pm\sqrt{-1 - B_{\vec{n}}^{(1)} C_{\vec{n}}^{(1)}}. \tag{27}$$

Note that $B_{\vec{n}}^{(1)} C_{\vec{n}}^{(1)} \leq -1$, since $A_{\vec{n}}^{(1)}$ must be a real number.

Once we have deduced the general expression of the invariant complex structures in the basis $\{(q_{\vec{n}}, x_{\vec{n}}, p_{\vec{n}}, y_{\vec{n}})\}$, it is straightforward to write them in the basis $\{(a_{\vec{n}}, a_{\vec{n}}^*, \tilde{a}_{\vec{n}}, \tilde{a}_{\vec{n}}^*)\}$ of annihilation and creationlike variables defined by the complex structure J_0, basis in which we can easily compare them. The considered change of basis can be obtained from Eqs. (7). In this way, one can check that the blocks $J_{\vec{n}}$ of any admissible invariant complex structure must be block diagonal, with 2×2 blocks that coincide by pairs. More explicitly,

$$J_{\vec{n}} = \begin{pmatrix} \mathcal{J}_{\vec{n}} & \mathbf{0}_{2\times 2} \\ \mathbf{0}_{2\times 2} & \mathcal{J}_{\vec{n}} \end{pmatrix}, \tag{28}$$

with

$$\mathcal{J}_{\vec{n}} = i \begin{pmatrix} -B_{\vec{n}}^{(1)} + C_{\vec{n}}^{(1)} \omega_n^2 & -B_{\vec{n}}^{(1)} - C_{\vec{n}}^{(1)} \omega_n^2 + i A_{\vec{n}}^{(1)} \omega_n \\ B_{\vec{n}}^{(1)} + C_{\vec{n}}^{(1)} \omega_n^2 + i A_{\vec{n}}^{(1)} \omega_n & B_{\vec{n}}^{(1)} - C_{\vec{n}}^{(1)} \omega_n^2 \end{pmatrix}. \tag{29}$$

Actually, every invariant complex structure J (that is compatible with the symplectic structure) can be obtained from J_0 by means of a symplectic transformation, K, namely

$J = KJ_0K^{-1}$ [16]. Since, in the considered basis of annihilation and creationlike variables, J_0 is diagonal with blocks of the form $(J_0)_{\vec{n}} = \text{diag}\{i, -i, i, -i\}$, the transformation K can also be taken block diagonal, with 4×4 blocks of the type

$$K_{\vec{n}} = \begin{pmatrix} \mathcal{K}_{\vec{n}} & \mathbf{0}_{2\times 2} \\ \mathbf{0}_{2\times 2} & \mathcal{K}_{\vec{n}} \end{pmatrix}, \qquad \mathcal{K}_{\vec{n}} = \begin{pmatrix} \kappa_{\vec{n}} & \lambda_{\vec{n}} \\ \lambda^*_{\vec{n}} & \kappa^*_{\vec{n}} \end{pmatrix}. \tag{30}$$

Here, $\kappa_{\vec{n}}$ and $\lambda_{\vec{n}}$ are complex numbers which play the role of alpha and beta Bogoliubov coefficients for the transformation K. In particular, they satisfy the symplectomorphism condition

$$|\kappa_{\vec{n}}|^2 - |\lambda_{\vec{n}}|^2 = 1 \qquad \forall \vec{n}. \tag{31}$$

Finally, the relation of the coefficients $\kappa_{\vec{n}}$ and $\lambda_{\vec{n}}$ with the matrix elements $A^{(1)}_{\vec{n}}$, $B^{(1)}_{\vec{n}}$, and $C^{(1)}_{\vec{n}}$ is given by

$$2|\kappa_{\vec{n}}|^2 = 1 - B^{(1)}_{\vec{n}} + C^{(1)}_{\vec{n}}\omega_n^2, \tag{32}$$

$$2\kappa_{\vec{n}}\lambda_{\vec{n}} = B^{(1)}_{\vec{n}} + C^{(1)}_{\vec{n}}\omega_n^2 - iA^{(1)}_{\vec{n}}\omega_n. \tag{33}$$

Notice that the phase of $\kappa_{\vec{n}}$ can be chosen freely. For instance one can choose it so that this Bogoliubov coefficient be non-negative. Actually, one can see that this choice does not affect the rest of our considerations.

3.3 Uniqueness of the invariant representation with unitary dynamics

To conclude our proof, showing the validity of our criterion to pick out a unique Fock representation, we still have to consider the unitary implementation of the dynamics and demonstrate that this restricts the admissible invariant complex structures to only one class of unitary equivalence. We will use again the fact that all the possible invariant complex structures J are related with the complex structure J_0 by means of a symplectomorphism K of the form given above, which can be understood just as a change of annihilation and creationlike variables. It follows that the unitary implementation of the evolution U in the representation determined by J amounts to the unitary implementation of $K^{-1}UK$ with respect to J_0 [16]. The beta coefficients of $K^{-1}UK$ can be viewed as the antilinear coefficients of the Bogoliubov transformation determined by the dynamics, expressed in terms of the annihilation and creationlike operators selected by J, instead of by J_0. A trivial computation shows that these new beta coefficients, that we will call $\beta^J_{\vec{n}}$, take the following expression in terms of the original Bogoliubov coefficients:

$$\beta^J_{\vec{n}}(t, t_0) = (\kappa^*_{\vec{n}})^2\beta_n(t, t_0) - (\lambda_{\vec{n}})^2\beta^*_n(t, t_0) + 2i\kappa^*_{\vec{n}}\lambda_{\vec{n}}\Im[\alpha_n(t, t_0)]. \tag{34}$$

Here, $\Im[\cdot]$ denotes the imaginary part. It is worth remarking that these beta coefficients (or rather beta functions, taking into account the time variation) depend now not just on n, the label of the eigenspaces of the LB operator, but rather on $\vec{n}$, which is the label of the sine and cosine modes and, as a consequence, of the "irreducible" *real* representations

of the symmetry group of the three-torus. Therefore, if, according to our criterion, we restrict our discussion exclusively to invariant complex structures J which allow a unitary quantum evolution, the above beta functions will have to be square summable (over $\vec{n}$) at all values of time $t \in \mathbb{I}$. So, we will assume that this is the case from now on.

The analysis of this summability can be made along a line of arguments similar to that presented in Ref. [21]. Using that the beta functions $\beta_n(t,t_0)$ corresponding to J_0 are square summable and that $|\kappa_{\vec{n}}| \geq 1$, one can prove that the summability (in square norm) of $\beta^J_{\vec{n}}(t,t_0)$ implies the same property for the set formed by $\Im[\alpha_n(t,t_0)]\lambda_{\vec{n}}/\kappa^*_{\vec{n}}$. Recalling then the asymptotic behavior of $\alpha_n(t,t_0)$ and calling $z_{\vec{n}} = \lambda_{\vec{n}}/\kappa^*_{\vec{n}}$, we arrive at the conclusion that the quantities

$$z_{\vec{n}} \sin\left[\omega_n(t-t_0) + \frac{1}{2\omega_n}\int_{t_0}^{t} d\bar{t}\, s(\bar{t})\right] \tag{35}$$

form a set which is square summable. The deduction of this result assumes, as a sufficient (but not necessary) condition, that the mass function $s(t)$ possesses a second derivative which is integrable in every compact subinterval of the time domain $\mathbb{I}$. A time integration, over any such subinterval, of the partial sums of the square norms of the elements (35), combined with a suitable application of Luzin's theorem [32] (which is possible because the considered elements are measurable functions), shows then that the set formed by $z_{\vec{n}}$ (namely, the ratios of the coefficients of $\mathcal{K}_{\vec{n}}$) is square summable. Given the definition of $z_{\vec{n}}$ and relation (31), one can see that this implies that the set formed by the antilinear coefficients of K, $\lambda_{\vec{n}}$, is square summable as well [21]. But this last summability result is precisely the condition for the unitary implementability of the symplectomorphism K and, hence, of the unitary equivalence of the two Fock representations determined the complex structures J and J_0, related by that symplectomorphism. Since the discussion is valid for all admissible invariant complex structures J, we conclude that all such structures which besides allow for a unitary dynamics are indeed equivalent. Thus, our criterion of invariance under the three-torus isometries and of unitarity in the evolution picks out a unique equivalence class of (invariant) representations for the scalar field.

4. Uniqueness of the field description

We will show now that our criterion is not only capable of selecting a unique equivalence class of Fock representations for the quantization of a KG field with a time dependent mass in a flat spacetime with the spatial topology of a three-torus, but, beyond that, it also determines a unique canonical pair for the field among all those that are related by a linear canonical transformation varying in time and in which the field configuration gets scaled. This kind of scaling transformations are often found in cosmological contexts, either when dealing with test fields or with perturbations around homogeneous and isotropic solutions, which can represent a genuine background spacetime, an effective spacetime on which the propagation takes place once certain quantum effects are taken into account, or just an auxiliary background in which one formulates in a simpler

form the field dynamics (for instance after dimensional reduction in symmetric models in General Relativity). The scaling absorbs part of the time dependence of the field, which is assigned to the time variation of the background. Although the classical formulations obtained for the field with this class of transformations are all equivalent, this ceases to be the case in the quantum theory, both because not all linear canonical transformations admit a unitary implementation and because the time dependence of the transformation changes the dynamics, in particular affecting its properties of unitarity. As a consequence, the criterion of a unitary evolution has different implications in the distinct formulations reached with these transformations.

4.1 Time dependent canonical transformations

The most general linear canonical transformation which includes a scaling of the field configuration and allows for a time dependence of the linear coefficients has the form

$$\phi = f(t)\varphi, \qquad P_\phi = \frac{P_\varphi}{f(t)} + g(t)\varphi. \tag{36}$$

In the last equation, we have taken again into account that the determinant of the three-torus metric is equal to one. We assume that the real functions $f(t)$ and $g(t)$ that characterize the transformation are at least twice differentiable, in order to respect the differentiability properties in the field equations. Besides, we suppose that the function $f(t)$ never vanishes, so that no spurious singularity is introduced in the field with the considered transformation. Finally, by means of a constant canonical linear transformation (which does not change the Fock representation of the field system), we can always set the initial values of the two time functions involved in our change equal to $f(t_0) = 1$ and $g(t_0) = 0$ [18], so that the original and the transformed canonical pairs coincide initially.

For the new pair (ϕ, P_ϕ), we start adopting the representation determined by the complex structure J_0. The time dependent transformation which leads to this pair changes the dynamics with respect to the original one, U. The new dynamical transformation $\tilde{U}$ has 2×2 blocks labeled again by the LB eigenvalue number n, and given by [23]:

$$\tilde{U}_n(t, t_0) = C_n(t) U_n(t, t_0), \tag{37}$$

where

$$C_n(t) := \begin{pmatrix} f_+(t) + iG_n(t) & f_-(t) + iG_n(t) \\ f_-(t) - iG_n(t) & f_+(t) - iG_n(t) \end{pmatrix}, \tag{38}$$

$$2f_\pm(t) := f(t) \pm \frac{1}{f(t)}, \qquad G_n(t) = \frac{g(t)}{2\omega_n}. \tag{39}$$

On the other hand, let us recall that the most general invariant complex structure J, compatible with the symplectic structure, has already been characterized in Sec. 3.2: it is related with J_0 by a symplectomorphism K of the type (30). In total, then, we

see that the new dynamics $\tilde{U}$ turns out to admit a unitary implementation with respect to an invariant complex structure J if and only if the beta Bogoliubov functions of the transformations with blocks $\mathcal{K}_{\vec{n}}^{-1}C_n(t)U_n(t,t_0)\mathcal{K}_{\vec{n}}$ are square summable over all the possible values of $\vec{n}$ (the label of the sine and cosine modes). These beta functions, that we will call $\tilde{\beta}^J_{\vec{n}}(t,t_0)$, adopt an expression similar to that given in Eq. (34), but with the Bogoliubov functions $\alpha_n(t,t_0)$ and $\beta_n(t,t_0)$ –corresponding to the reference complex structure J_0– replaced with those of the evolution $\tilde{U}$ for the pair (ϕ, P_ϕ) (see Ref. [23]):

$$\tilde{\alpha}_n(t,t_0) = f_+(t)\alpha_n(t,t_0) + f_-(t)\beta_n^*(t,t_0) + iG_n(t)[\alpha_n(t,t_0) + \beta_n^*(t,t_0)], \tag{40}$$

$$\tilde{\beta}_n(t,t_0) = f_+(t)\beta_n(t,t_0) + f_-(t)\alpha_n^*(t,t_0) + iG_n(t)[\beta_n(t,t_0) + \alpha_n^*(t,t_0)]. \tag{41}$$

In the rest of this section, we will demonstrate that a unitary dynamics is possible only if $f(t) = 1$ and $g(t) = 0$ at all times, that is, if we describe our field precisely with the original canonical pair, associated with the formulation as a KG field in flat spacetime with compact spatial sections and a time dependent mass.

4.2 Uniqueness of the scaling

We will first show that $f(t)$ must be the identity function in $\mathbb{I}$. In order to do this, we simply adapt the proof explained in Ref. [26]. For each eigenvalue of the LB operator, $-\omega_n^2$, let us choose a value $\vec{M}_n$ of the label $\vec{n}$ among all those whose Euclidean norm as a vector coincides with ω_n. We then consider the sequence with elements $\tilde{\beta}^J_{\vec{M}_n}(t,t_0)$. This sequence is a subset of the beta functions $\tilde{\beta}^J_{\vec{n}}(t,t_0)$, obtained by ignoring the degeneracy of the LB eigenspaces. Since the latter set is square summable if we admit that the dynamics is unitary, then, a fortiori, the sequence with labels $\vec{M}_n$ is also square summable at all times. Recalling the asymptotic behavior (12) and employing that $|\kappa_{\vec{M}_n}| \geq 1$, it is not difficult to check that the considered square summability implies a vanishing limit at all times for the sequence with terms

$$\left[e^{i\omega_n(t-t_0)} - z^2_{\vec{M}_n} e^{-i\omega_n(t-t_0)}\right] f_-(t) - 2iz_{\vec{M}_n}\sin[\omega_n(t-t_0)]f_+(t). \tag{42}$$

Let us introduce now the real and imaginary parts of $z_{\vec{M}_n}$:

$$z_{\vec{M}_n} = \Re_{\vec{M}_n} + i\Im_{\vec{M}_n}. \tag{43}$$

A straightforward computation shows that a necessary condition for the vanishing limit of the (complex) sequence (42) is that the sequence

$$f_-\left(\Re^2_{\vec{M}_n} + \Im^2_{\vec{M}_n} - 1\right)[(1 + \Re^2_{\vec{M}_n} + \Im^2_{\vec{M}_n})f_- - 2\Re_{\vec{M}_n} f_+] \tag{44}$$

vanish as well in the limit $n \to \infty$, at all times [to simplify the notation, we have obviated the explicit time dependence of the functions $f_\pm(t)$]. On the other hand, detailed arguments developed in Ref. [26] demonstrate that a further necessary condition for the unitary implementability of the dynamics is that the sequence of elements $(\Re^2_{\vec{M}_n} + \Im^2_{\vec{M}_n} - 1)$ does not tend to zero.

The last step in our proof is to show that, then, the unitary implementation is not possible unless the function $f(t)$ is the unit function. Let us suppose just the opposite, namely, that $f(t)$ is not identically the unit function. Hence, at certain values of the time t, we will have that $f(t) \neq 1$. We will focus our discussion on those values of t and see that we arrive in fact at a contradiction. Notice that in the points that we are considering, we get $f_-(t) \neq 0$. Besides, recall that $f(t)$ is a positive and continuous function (actually, we have assumed that it is twice differentiable).

Returning to expression (44), a necessary condition for the unitary implementation of the dynamics is that the sequence with elements

$$(\Re^2_{\vec{M}_n} + \Im^2_{\vec{M}_n} - 1)[(1 + \Re^2_{\vec{M}_n} + \Im^2_{\vec{M}_n})f_- - 2\Re_{\vec{M}_n} f_+] \tag{45}$$

tends to zero at all the instants of t under consideration. In addition, since we know that the sequence formed by $(\Re^2_{\vec{M}_n} + \Im^2_{\vec{M}_n} - 1)$ cannot tend to zero at large n [26], we can assure that there exists a number $\epsilon > 0$ and a subsequence S of positive integers n such that $|\Re^2_{\vec{M}_n} + \Im^2_{\vec{M}_n} - 1| > \epsilon$ in S. Clearly, this fact implies that the second factor that appears in Eq. (45) must have a vanishing limit on that subsequence. Using this result we straightforwardly deduce that the following expression must have a zero limit on the studied subsequence S:

$$f^2(t)[(1 - \Re_{\vec{M}_n})^2 + \Im^2_{\vec{M}_n}] - [(1 + \Re_{\vec{M}_n})^2 + \Im^2_{\vec{M}_n}]. \tag{46}$$

But, given that the two time independent sequences $(1-\Re_{\vec{M}_n})^2+\Im^2_{\vec{M}_n}$ and $(1+\Re_{\vec{M}_n})^2+\Im^2_{\vec{M}_n}$ cannot both tend to zero, the vanishing of the limit of the above expression immediately requires that the function $f(t)$ take exactly the same value at at all the instants of time that we are considering (namely, those where $f(t) \neq 1$). In this way, we reach the conclusion that the function $f(t)$ can take at most two distinct values: one of them equal to 1 (e.g., at the reference time t_0) and maybe another value which has been assumed to be different from the unity. However, such a behavior is precluded by the continuity of the function. This clear contradiction proves that the only consistent possibility is that $f(t)$ is indeed identically equal to the unit function, as we wanted to demonstrate.

In this way, we reach the result that no scaling of the field configuration is allowed by our combined criterion of invariance under the three-torus isometries and the unitarity of the dynamics.

4.3 Uniqueness of the field momentum

We will end this section by proving that no change in the momentum is permitted by the requirement of symmetry invariance and unitary evolution, so that the function $g(t)$ must vanish. We return to the expression of the beta functions for the dynamics of the system after performing a time dependent linear canonical transformation, but now specialized to the case $f(t) = 1$, in accordance with the discussion of the previous subsection. The demand that, for an invariant complex structure J, the dynamics admit a unitary implementation amounts to the square summability of the set formed by $\tilde{\beta}^J_{\vec{n}}(t, t_0)$ at all

instants of time. Recalling that $|\kappa_{\vec{n}}| \geq 1$, this summability ensures the same property for the set given by $\tilde{\beta}^J_{\vec{n}}(t, t_0)/(\kappa^*_{\vec{n}})^2$. Then, using the asymptotic relations (12), and that $|z_{\vec{n}}| \leq 1$, one can deduce that the set formed by

$$G_n(t)\left\{e^{i[\omega_n(t-t_0)-\delta_{\vec{n}}]} + |z_{\vec{n}}|^2 e^{-i[\omega_n(t-t_0)-\delta_{\vec{n}}]} + 2|z_{\vec{n}}|\cos[\omega_n(t-t_0)]\right\}$$
$$+2|z_{\vec{n}}|\Im\left[\alpha_n(t,t_0)\right] \tag{47}$$

is square summable at all times in the considered interval $\mathbb{I}$. Here, $\delta_{\vec{n}}$ is the phase of $z_{\vec{n}}$. Obviously, the square summability is also true for the set obtained by dividing those terms by ω_n, since this eigenvalue provides a sequence that diverges to infinity. Hence, employing that g_n/ω_n^4 is a summable sequence (as we proved in Sec. 3.1) and the definition of $G_n(t)$, we conclude that the set with elements $|z_{\vec{n}}|\Im\left[\alpha_n(t,t_0)\right]/\omega_n$ must also be square summable. By performing a convenient time average and making use of Luzin's theorem, one can show then that the set formed by $|z_{\vec{n}}|/\omega_n$ has to be square summable as well.

Taking into account this result in the consideration of the terms (47), we arrive at the square summability of the set formed by

$$G_n(t)e^{i[\omega_n(t-t_0)-\delta_{\vec{n}}]} + 2|z_{\vec{n}}|\Im\left[\alpha_n(t,t_0)\right]. \tag{48}$$

In particular, the imaginary part of these quantities, namely

$$\frac{g(t)}{2\omega_n}\sin\left[\omega_n(t-t_0) - \delta_{\vec{n}}\right] \tag{49}$$

[where we have used again the definition of $G_n(t)$], is also a square summable set at all times. If there existed a subinterval of $\mathbb{I}$ where the function $g(t)$ did not vanish, a suitable time integration over it would lead to the conclusion that the sequence with elements g_n/ω_n^2 must be summable [26]. But this sequence has in fact a divergent sum, because the sum exceeds that of $D_N/(N+1)^2$ over the positive integers, which clearly diverges, given the asymptotic behavior $D_N \propto N^2$ discussed in Sec. 3.1. This eliminates the possibility that $g(t)$ may differ from zero in any subinterval of $\mathbb{I}$. Since the function is continuous, this implies that $g(t)$ has to be the zero function.

Summarizing, our criterion determines completely the choice of canonical pair for the field among all the possibilities related by means of a time dependent linear canonical transformation. The criterion removes the freedom to scale the field, and to redefine its momentum by including a contribution that is linear in the field configuration.

5. Conclusions

We have analyzed two types of ambiguities that appear in the Fock quantization of scalar fields in cosmological spacetimes with spatial sections that are isomorphic to a three-torus. The first of these ambiguities is related to the possibility of scaling the field by means of a time dependent function, which assigns to the (physical, effective or auxiliary) background cosmological spacetime part of the time variation. This scaling

can be viewed as resulting from a linear canonical transformation, in which the field momentum gets the inverse scaling (compared to the field configuration). In addition, the momentum may admit a linear contribution of the field configuration, with a time dependent coefficient. Canonical transformations of this kind often lead to a simpler and better behaved formulation for the system [5]. Each of these transformations provides a different canonical pair for the field description and, furthermore, changes the dynamics, since part of the time dependence is absorbed in the background. The other ambiguity that we have considered refers to the possible choices of Fock representation for the CCR's, once a specific canonical pair (and dynamical evolution) is given for the field. The physically different representations can be understood as corresponding to inequivalent choices of vacuum for the Fock construction. Alternatively, the physical freedom in the selection of a representation can be assigned to the possible inequivalent choices of complex structure. This ambiguity is well known in quantum field theory [2], and it is common to remove it by introducing certain requirements on the vacuum, or equivalently on the complex structure, such as incorporating certain symmetries of the background spacetime, or presenting a especially good local or dynamical behavior.

We have put forward a criterion to remove these ambiguities in systems that, by means of one of the considered time dependent canonical transformations, can be formulated as a KG field with time varying mass propagating in a flat spacetime with (compact) three-torus spatial topology. The criterion consists of two requirements. First, that the complex structure (and hence the vacuum) be invariant under the isometries of the three-torus, equipped with the standard metric. Second, that this complex structure also allows for the unitary implementation of the dynamics. Notice, in particular, that the considered ambiguity under time dependent canonical transformations affects the dynamical evolution of the system, so that our demand of unitarity has different implications for the distinct field descriptions obtained with such transformations. On the other hand, requiring a unitary dynamics guarantees that the standard probabilistic interpretation of quantum field theory is consistent in the evolution, so that one does not have to renounce to it.

We have demonstrated that our criterion is indeed capable of removing the two mentioned types of ambiguities, selecting a unique canonical pair for the field among all those related by linear canonical transformations with time dependent coefficients and, furthermore, selecting a unique class of unitarily equivalent Fock representations for the corresponding CCR's. This class contains the representation that would be naturally associated with the case of a massless field, though now employed to define a Fock quantization of a system that is not only massive, but whose mass changes in time. To obtain this uniqueness result, we have assumed only a very mild restriction on the time dependent mass: that it must possess a second derivative which is integrable in every compact subinterval of the time domain. In fact, this assumption is just a sufficient condition, but does not even seem to be strictly a necessary one. Let us also emphasize that the conclusion about the uniqueness of the Fock quantization is valid for *any* possible time domain provided that it is a (non-infinitesimal) interval of the real line.

Our result about the uniqueness of the Fock quantization can be applied to a number of physically interesting situations in cosmology. Current observations indicate that the large scale structure of the universe is (approximately) homogeneous and isotropic. Therefore, (quantum) matter fields in cosmology are naturally described by quantum field theory in FRW spacetimes. Besides, inhomogeneities can then be treated as perturbations in a fairly good approximation [33]. At leading order, these perturbations are also described as linear fields that propagate in FRW spacetimes. Furthermore, the observations favor a spatially flat cosmology, which is precisely the case studied in this work. On the other hand, the fact that the spatial topology is assumed to be compact should not pose a severe restriction, because one would expect that, beyond a certain cosmological scale related with the Hubble radius, physical interactions should have no relevant effect. Then, the physics would not be altered importantly by considering a compactification scale larger than this cosmological one. In this general context, a simple system that can be reformulated by means of a scaling as a KG field with time dependent mass in a flat spacetime is a massive, minimally coupled scalar field. A description of this type is also obtained after a suitable scaling for cosmological perturbations in conformal time. For instance, this is the case of tensor perturbations (for which one can generalize our considerations, presented here for scalar fields) and of the gauge invariant energy density perturbation amplitude [5, 34]. In addition, one obtains a similar kind of field description for scalar perturbations of a massive scalar field around flat FRW spacetimes after adopting (e.g.) a longitudinal gauge, with the only caveat that the KG equation is modified with subdominant terms which, nonetheless, do not affect the asymptotic behavior employed in our discussion [8].[3] The fact that, in all these cases, the criterion of spatial symmetry invariance and unitary evolution turns out to determine a unique Fock quantum description provides the quantum field theory and its predictions with the desired robustness.

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Proper Time and Length in Schwarzschild Geometry

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Abstract: We study proper time (τ) intervals for observers at rest in the universe (U) and anti-universe ($\bar{U}$) sectors of the Kruskal-Schwarzschild eternal spacetime of mass M, and proper lengths (ρ) in the black hole (BH) and white hole (WH) sectors. The fact that in asymptotically flat regions, coordinate time t at infinity is proper time, leads to a past directed Kruskal time T in $\bar{U}$. In the BH and WH sectors maximal proper lengths coincide with maximal proper time intervals, πM, in these regions, i.e. with the proper time of radial free falling (ejection) to (from) the singularity starting (ending) from (at) rest at the horizon.

Keywords: Proper Time; Proper Length; Schwarzschild Spacetime; Kruskal Diagram
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1. Proper Times in U and $\bar{U}$

All our analysis will be based on the Kruskal diagram corresponding to Schwarzschild spacetime i.e. in its maximal analytical extension [1]. This is illustrated in Figure 1. We use geometric units $G_N = c = 1$.

In region $I = U$ (universe), $r > 2M$, the square of the proper time is given by

$$d\tau^2 = (1 - \frac{2M}{r})dt^2 - \frac{dr^2}{1 - \frac{2M}{r}} - r^2 d\Omega^2 \qquad (1)$$

where M is the mass and $d\Omega^2 = d\theta^2 + sin^2\theta d\varphi^2$. For $r \to \infty$ spacetime is flat and t is

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the proper time. For an observer at rest at $r_0 > 2M$,

$$d\tau^2 = (1 - \frac{2M}{r_0})dt^2 \tag{2}$$

and therefore

$$\Delta\tau_U = \sqrt{1 - \frac{2M}{r_0}}\Delta t, \quad \Delta t = t_2 - t_1. \tag{3}$$

I.e. $\Delta\tau_U = \Delta\tau_U(M; r_0, \Delta t)$; so, $\Delta\tau_U \to 0$ as $r_0 \to 2M$ (time "does not pass" for light) and $\Delta\tau_U \to \Delta t$ as $r_0 \to \infty$. The maximum value of $\Delta\tau_U$ is $+\infty$ since this is the maximum value of Δt.

To determine the corresponding $\Delta\tau_{\bar{U}}$ (by symmetry it should equal $\Delta\tau_U$) we have to use the expression for $d\tau^2$ in terms of the Kruskal variables T and R, valid in the four regions U, BH, $\bar{U}$, and WH [2]:

$$d\tau^2 = 4 \times \frac{2M}{r}\, e^{-\frac{r}{2M}}\, (dT^2 - dR^2 - r^2 d\Omega^2), \tag{4}$$

with r implicitly given in terms of T and R by

$$\frac{1}{2M^2}(R^2 - T^2) = (\frac{r}{2M} - 1)e^{\frac{r}{2M}}. \tag{5}$$

In region $I' = \bar{U}$ (anti-universe) the relation between T and R with t and r is given by

$$T = -2M\sqrt{\frac{r}{2M} - 1}\, e^{\frac{r}{4M}}\, Sh(\frac{t}{4M}) \in (-\infty, +\infty), \tag{6}$$

and

$$R = -2M\sqrt{\frac{r}{2M} - 1}\, e^{\frac{r}{4M}}\, Ch(\frac{t}{4M}) \in (-\infty, 0). \tag{7}$$

For constant r,

$$dT = \frac{\partial T}{\partial t}dt = -\frac{1}{2}\sqrt{\frac{r}{2M} - 1}\, e^{\frac{r}{4M}}\, Ch(\frac{t}{4M})dt,$$
$$dR = \frac{\partial R}{\partial t}dt = -\frac{1}{2}\sqrt{\frac{r}{2M} - 1}\, e^{\frac{r}{4M}}\, Sh(\frac{t}{4M})dt$$

and therefore

$$dT^2 - dR^2 = ((\frac{\partial T}{\partial t})^2 - (\frac{\partial R}{\partial t})^2)dt^2 = \frac{1}{4}(\frac{r}{2M} - 1)\, e^{\frac{r}{2M}}\, dt^2. \tag{8}$$

Then, $d\tau^2_{\bar{U}}(r_0) = (1 - \frac{2M}{r_0})dt^2$ and so

$$d\tau_{\bar{U}}(r_0) = \pm\sqrt{1 - \frac{2M}{r_0}}dt. \tag{9}$$

The minus sign would lead to $d\tau_{\bar{U}}(r_0) < 0$ for $dt > 0$ or viceversa, $d\tau_{\bar{U}}(r_0) > 0$ for $dt < 0$. No of these results is admisible, since both τ (at finite distances) and t (at infinity) are

proper times, and *any proper time must always be future directed* [3]. Then, the plus sign has to be chosen in (9), and therefore

$$\Delta\tau_{\bar{U}}(r_0) = \sqrt{1 - \frac{2M}{r_0}}\Delta t = \Delta\tau_U(r_0). \tag{10}$$

But then $T_2 < T_1$ i.e. *Kruskal time decreases*:

$$\Delta T(r_0) = T_2 - T_1 = -2M\sqrt{\frac{r_0}{2M} - 1}\; e^{\frac{r_0}{4M}}\; (Sh(\frac{t_2}{4M}) - Sh(\frac{t_1}{4M}) < 0 \tag{11}$$

since $t_2 > t_1$. The fact that T is *past directed* in $\bar{U}$, indicates that T is not a physical time in $\bar{U}$, but only a coordinate (though global) in Kruskal spacetime, with no intrinsic physical meaning, at least with respect to the eternal black hole.

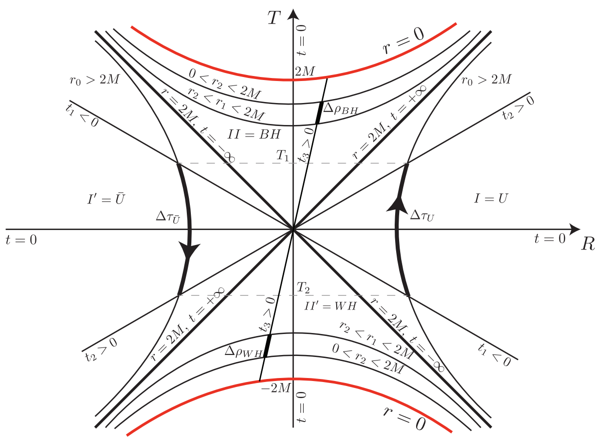

Fig. 1 Proper time and length in Kruskal diagram.

2. Proper Lengths in BH and WH

In region $II = BH$ (black hole), $0 < r < 2M$, the square of the interval is given by

$$ds^2 = \frac{dr^2}{\frac{2M}{r} - 1} - (\frac{2M}{r} - 1)dt^2 - r^2 d\Omega^2 \tag{12}$$

which, at fixed θ, φ, and time t_3 can be interpreted as the *elementary proper length* $d\rho$ along dr:

$$d\rho_{BH} = \frac{dr}{\sqrt{\frac{2M}{r} - 1}}, \tag{13}$$

independent of t_3. Integrating this expression between r_2 and r_1 gives

$$\Delta\rho_{BH}(r_2, r_1) = \int_{r_2}^{r_1} dr\sqrt{\frac{r}{2M - r}} = 2M\int_{x_2}^{x_1} dx\sqrt{\frac{x}{1 - x}} \tag{14}$$

with $x = \frac{r}{2M}$ and $x_i = \frac{r_i}{2M}$, $i = 1, 2$. Using

$$\int dx\sqrt{\frac{x}{1 - x}} = -\sqrt{x(1 - x)} + arctg(\frac{\sqrt{x(1 - x)}}{1 - x}) + const. \tag{15}$$

one obtains, in particular for the limits $r_2 \to 0_+$ and $r_1 \to (2M)_-$,

$$\Delta\rho_{BH}(0, 2M) = 2\pi \times \frac{\pi}{2} = \pi M. \tag{16}$$

So, the *maximal proper length* in BH coincides with the proper time of radial free falling to the future singularity at $r = 0$ of a massive test particle, starting at rest from the future horizon [4].

By symmetry, the same result should hold in region $II' = WH$ (white hole), but now the maximal proper length should coincide with the proper time of radial free ejection from the past singularity at $r = 0$ of a massive test particle, ending at rest at the past horizon. In fact, in II' the relation between T and R with t and r is given by

$$T = -2M\sqrt{1 - \frac{r}{2M}}\; e^{\frac{r}{4M}}\; Ch(\frac{t}{4M}) \in (-\infty, 0), \tag{17}$$

and

$$R = -2M\sqrt{1 - \frac{r}{2M}}\; e^{\frac{r}{4M}}\; Sh(\frac{t}{4M}) \in (-\infty, +\infty). \tag{18}$$

For constant t,

$$dT = \frac{\partial T}{\partial r}dr = \frac{r}{4M}\; \frac{e^{\frac{r}{4M}}}{\sqrt{1 - \frac{r}{2M}}}\; Ch(\frac{t}{4M})dt,$$

$$dR = \frac{\partial R}{\partial r}dr = \frac{r}{4M}\; \frac{e^{\frac{r}{4M}}}{\sqrt{1 - \frac{r}{2M}}}\; Sh(\frac{t}{4M})dr,$$

and using again (4) with $d\theta = d\varphi = 0$, one obtains

$$d\rho_{WH} = d\rho_{BH} \tag{19}$$

which, after integration, leads to the same results (14) and (16), but for $\Delta\rho_{WH}$.

3. Final Comment

It is believed that, probably, eternal black holes do not exist in nature, and that only black holes resulting from gravitational collapse (and also primordial black holes produced in the very early universe) exist [5],[6]. For black holes produced in gravitational collapse, T behaves as a physical time coordinate since, as proper (τ) and coordinate (t) times, it also is future directed. Nevertheless, eternal black holes are solutions of Einstein equations, and for them, as shown in section 1, T looses physical character in region $\bar{U}$.

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A Fisher-Bohm Geometry for Quantum Information

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Abstract: The underlying geometry of Bohms quantum potential is explored as a new approach to quantum information. If we express the entropy of a quantum system as a superposition of Boltzmann entropies, Bohms quantum potential, under the constraint of a minimum condition of Fisher information, appears related to a Weyl-like gauge potential and the quantum information emerges as a deformation of the quantum system geometry, according to B. Hiley and the etymological meaning of information: the activity of shaping or putting form into a given process. For this purpose, it will be important to study the quantum potential as an expression of quantum entropy in the light of the metric of Fisher-Weyl. Finally, we will examine the geometry of the double-slit interference and the Aharonov-Bohm effect.

Keywords: Quantum Potential; Fisher Information; Quantum Geometry; Weyl-like gauge; Double Slit Interferometer ; Aharonov-Bohm effect
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1. Introduction

In the context of classical mechanics, once a classical variable (for examples position or mass) has been measured, it is correct to assume that further measurements will not provide any new information. In the case of the localization, for example, the object will continue to be at its position owing to the fact that the object can, in principle, be isolated from the environment.

In quantum mechanics, the situation is more complex, because the quantum state could be pure or mixed, and these two cases are very different from the point of view of measurement. Moreover, in general, one does not know whether the state is pure or

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mixed. The game for the receiver is to determine this state as closely as possible, after examining as many copies of the state as it is required. The entropy can be thus sees as the average information communicated about the unknown state at any point in the measurement process.

A mixed state is a statistical mixture of component pure states, and its entropy is determined by the von Neumann measure in a manner that is similar to the entropy for classical states. A pure state is completely described by its state function and its von Neumann entropy is zero.

Several theories have been advanced that assign a finite entropy to matter (see, for example, the fundamental work of J. Bekenstein [1]). The finite value of entropy for a given volume has been taken to mean that matter cannot be subdivided infinitely, and that the fundamental entity relating to matter is a bit (1 or 0) of information. However, this approach of discretization has not been very successful. Part of the fault may lie in the limitations of the current concept of quantum entropy. In particular, von Neumann's definition of entropy does not provide the right measure in the asymmetric situation where the choice of the state itself carries information. Von Neumann's definition of quantum entropy seems to meet problems in the interpretation of quantum information as pure states. In this regard, an alternative approach has been proposed by S. Kak that includes thermodynamic information of the pure states[2].

In this article, we suggest a non-Euclidean geometry as the source of a quantum information that derives, on quantum level, from a quantum entropy which is defined as a vector of superposition of different Boltzmann entropies. In this approach, the difference between classical and quantum information is similar to the difference between Euclidean and non-Euclidean geometry in the parameter space determined by the quantum entropy. Superposition states characteristic of quantum mechanics are determined by the deformation of the geometry of the background and are associated with a vector of superposed entropies. In this way we try to reconcile the relational definition of von Neuman with the thermodynamics of Kak in a synthesis which characterizes the geometric deformation of quantum potential as a measure of quantum information, in the spirit of Einstein [3].

The structure of this article is the following: in chapter 2 we make some considerations about von Neumann and Kak approaches to quantum entropy. In chapter 3 we introduce the most important features of Bohm's quantum potential, and in chapter 4 analyze the quantum potential in terms of Fisher information and Weyl geometry based on a definition of quantum entropy as a superposition of different Boltzmann entropies. Finally, in chapters 5 and 6 we will apply the theoretical framework of quantum geometry developed to two physical problems, namely the double-slit interference and the Aharonov-Bohm effect.

2. About von Neumann's and Kak's Quantum Entropy

In order to speak about quantum information, one can assume that the source and the receiver use the same basis vectors for the representation and the measurement of the

states (this assumption is necessary to establish the baseline of the game between the source and the receiver). For a quantum system characterized by the density operator ρ, the average information the experimenter obtains in the repeated observations of the very many copies of an identically prepared mixed state is given by the von Neumann entropy

$$S_n(\rho) = \sum_x \lambda_x \ln \lambda_x \tag{1}$$

where λ_x are the eigenvalues of the density matrix associated with the system. In particular, in the case of the mixed state described by the matrix

$$\rho = \begin{bmatrix} p & 0 \\ 0 & 1-p \end{bmatrix}, \tag{2}$$

the entropy (1) becomes

$$S_n(\rho) = -p \ln p - (1-p) \ln(1-p). \tag{3}$$

Equation (3) implies that the von Neumann entropy of a pure state is zero, indicating that once it has been identified then there is no further information to be obtained from its copies, which is not the case with a mixed state. Since an unknown pure state will communicate real information to the receiver, the von Neumann entropy cannot be considered as a reasonable measure of quantum information as regards pure states.

Considering the information transfer problem from the point of view of the preparer of the state and the experimenter, it is clear that both mixed and pure states provide information to the experimenter. In this regard, a detailed analysis has been provided recently by Kak: "*For a two-component elementary mixed state, the most information in each measurement is one bit, and each further measurement of identically prepared states will also be one bit. For an unknown pure state, the information in it represents the choice the source has made out of the infinity of choices related to the values of the probability amplitudes with respect to the basis components of the receiver's measurement apparatus. The maximum information in a pure state is thus infinite. On the other hand, each measurement of a two-component pure state can provide one bit of information. But if it is assumed that the source has made available an unlimited number of identically prepared states, the receiver can obtain additional information from each measurement until the probability amplitudes have been correctly estimated. Once that has occurred, unlike the case of a mixed state, no further information will be obtained from testing additional copies of this pure state. The receiver can do this by adjusting the basis vectors so that he gets closer to the unknown pure state. As the adjustment proceeds, the amount of information that he would obtain from each measurement will decrease. The information that can be obtained from such a state in repeated experiments is potentially infinite in the most general case. But if the observer is told what the pure state is, the information associated with the states vanishes, suggesting that a fundamental divide exists between objective and*

subjective information. [...] One can speak of information associated with a system only in relation to an experimental arrangement together with the protocol for measurement. The experimental arrangement is thus integral to the amount of information that can be obtained." [2].

In order to conciliate the fact that, according to the von Neumann the entropy for an unknown pure state is zero with the fact that repeated measurements on copies of such a pure state do communicate information, Kak has proposed a measure for the informational entropy of a quantum state that includes information in the pure states and the thermodynamic entropy. In Kak's proposal the origin of information is explained in terms of an interplay between unitary and non-unitary evolution. More precisely, the starting idea of Kak's model is that the informational entropy of a quantum system with density matrix ρ is given by relation

$$S_i(\rho) = -\sum_i \rho_{ii} \ln \rho_{ii}. \tag{4}$$

The entropy (4) indicates the average uncertainty that the receiver has in relation to the quantum state *for each measurement.* Should the manner of the preparation of the pure state be known to the observer, he can choose a basis state function that would completely describe it, and there would indeed be no information associated with it.

By appropriately adjusting the basis vectors, the receiver can change the value of this entropy. The value of $S_i(\rho)$ is the amount of entropy of the quantum system that is accessible to the receiver.

Some properties of Si are the following:

1. $Si\ (\rho) \quad \geq Sn(\rho)$, and the two are equal only when the density matrix has only diagonal terms.

2. $Sn(\rho)$ is obtained by minimizing $Si\ (\rho)$ with respect to all possible unitary transformations.

3. The maximum value of Si is infinity, true for the case where the number of components is infinite.

It is interesting to remark that the informational entropy introduced by Kak can resolve the puzzle of entropy increase in the universe. It assumes that the universe had immensely large informational entropy namely was in a low-entropy quantum state in the beginning, and that then, during the physical evolution of the universe, this informational entropy was transformed into thermodynamic entropy and thus produced high-entropy quantum states, in part because of the second law of thermodynamics, in part because of the expansion of the universe, and above all because non-unitary evolutions of pure states. Given the fact that we have both unitary, U, and non-unitary, Mi , or measurement, operators, the density operator for each elementary state will change either to:

$$|\phi\rangle_{new} = \begin{cases} U|\phi\rangle \\ \frac{M_i|\phi\rangle}{\sqrt{\langle\varphi|M_i^+ M_i|\phi\rangle}} \end{cases} \tag{5}$$

where the first regards unitary evolution and the second non-unitary evolution When only non-unitary operators are used for the evolution, the elementary state will change from the pure state $|\rangle = \alpha\,|\rangle + \beta\,|\rangle$ to the mixed state given by the density matrix:

$$G = \begin{bmatrix} |\alpha|^2 & 0 \\ 0 & |\beta|^2 \end{bmatrix}. \tag{6}$$

Its informational entropy would then have transformed completely from that of the pure state to that of the mixed state, and its von Neumann entropy would now be finite.

The existence of non-unitary operators requires the presence of low-entropy structures. This view of the problem of how information increases is to postulate non-unitary evolution as a part of the earliest universe and requires the framework of dissipative quantum field theory [4].

3. Bohm's Quantum Potential as Active Information

In his classic papers of 1952 David Bohm showed that if one interprets each individual physical system as composed by a corpuscle and a wave guiding it, by writing its wave function in polar form and decomposing the Schrödinger equation, the movement of the corpuscle under the guide of the wave happens in agreement with a law of motion which assumes the following form

$$\frac{\partial S}{\partial t} + \frac{|\nabla S|^2}{2m} - \frac{\hbar^2}{2m}\frac{\nabla^2 R}{R} + V = 0, \tag{7}$$

where $R$is the amplitude and $S$is the phase of the wave function, $\hbar$ is Planck's reduced constant, $m$is the mass of the particle and $V$is the classical potential. This equation is the classical equation of Hamilton-Jacobi, except for the appearance of the additional term

$$Q = -\frac{\hbar^2}{2m}\frac{\nabla^2 R}{R}, \tag{8}$$

having the dimension of an energy, containing Planck constant and therefore appropriately defined quantum potential [5]. In analogous way, in the case of a many-body system, if we consider a wave function $\psi = R\left(\vec{x}_1, ..., \vec{x}_N, t\right) e^{iS(\vec{x}_1,...,\vec{x}_N,t)/\hbar}$, defined on the configuration space R^{3N} of a system of N particles, the movement of this system under the action of the wave ψ happens in agreement to the law of motion

$$\frac{\partial S}{\partial t} + \sum_{i=1}^{N} \frac{|\nabla_i S|^2}{2m_i} + Q + V = 0 \tag{9}$$

where

$$Q = \sum_{i=1}^{N} -\frac{\hbar^2}{2m_i}\frac{\nabla_i^2 R}{R} \tag{10}$$

is the many-body quantum potential.

In virtue of the features of the quantum potential, the basic equations (5) and (9) of non-relativistic Bohm theory do not imply a classical treatment of quantum processes. The non-locality emerges as a fundamental trait of quantum theory.In the expression of the quantum potential, the appearance of the amplitude of the wave function in the denominator also explains why the quantum potential can produce strong long-range effects that do not necessarily fall off with distance and so the typical properties of entangled wave functions. Thus even though the wave function spreads out, the effects of the quantum potential need not necessarily decrease This is just the type of behaviour required to explain the EPR paradox. In virtue of the quantum potential, Bohm's interpretation of quantum phenomena has the merit to include non-locality *ab initio* rather than to come upon it as an *a posteriori* statistical "mysterious weirdness".

Moreover, if we examine the expression of the quantum potential in the double-slit experiment, we find that it depends on the width of the slits, their distance apart and the momentum of the particle. This means that the quantum potential has a contextual nature, namely brings a global information on the process and its environment by individuating an infinite set of phase paths; and it has an active information in the sense that it modifies the behaviour of the particle [6]. In a double-slit experiment, if one of the two slits is closed the quantum potential changes, and this information arrives instantaneously to the particle, which behaves as a consequence. The active information of the quantum potential ((6) or (10)) is deeply different from classical one: it is, in fact, intrinsically not-Shannon-Turing computable[7, 8].

The quantum potential indicates a source of active information internal to the system and differently accessible according to the operations of preparation and state selection, environment and measurement. The quantum system's active information is defined by an infinite uncountable set of phase paths and its very nature is non-local. In such configurations as "quantum gates", based upon a generalization of Turing scheme, the constrains of reversibility and unitarity limit the possibility to detect quantum information just to the outputs of superposition states; and yet nothing prevents our thinking of a different approach to the system's geometry which, within peculiar experimental arrangements, can get qualitatively different answers and endowed with oracular skills, so turning into resource all non-locality features, even those which are traditionally regarded as a limit within the classical scheme, such as de-coherence, dissipation and probabilistic responses [9].

An essential key for the relations between quantum potential, system's geometry and information is provided by the Fisher information [10,11]. The Fisher information can be interpreted as the information an observable random variable X carries about a not-observable parameter θ which the probability distribution X depends on. Such statistical measurement has aroused interest in relation to the study of the distributions of the observables of a quantum system.

As for the physical meaning of the Fisher information, instead, there are very controversial viewpoints. Roy Frieden's programme [12] to derive physics' fundamental equations from an extreme physical Fisher information principle as optimization (or satu-

ration) of the observer/observed relationship has been widely criticized because of its vagueness. Actually, although Frieden's position is epistemologically correct for a good experimental physicist as he is, the principle itself is too less constraining, so that the significant physical features of the systems under observation have to be introduced *ad hoc* in order to make it really effective [13]. Thus, Frieden's programme looks more like a request for coherence between formal structures and distributions of observables than an out-and-out "fundamental principle". The studies where an attempt is made to connect Fisher information with the specific structural aspects of quantum mechanics and to consider it as a statistical indicator of the relationships between classical and quantum information seem much more interesting [14, 15, 16, 17, 18]. In spite of the "interpretative dilemmas", quantum mechanics shows the highest operational nature of any other physical theory, and it is thus greatly interesting that the "thin" statistical distribution of a quantum system can be derived from the quantum potential. In the next chapters, we will show that the Fisher information plays the role of a natural tile to build a metric able to connect the system's statistical outcomes and its global geometry.

4. The Quantum Potential as Fisher Metrics on Entropy Manifold

On the basis of an extension of the tensor calculus to operators represented by non-quadratic matrices [11, 21], it is possible to provide a new interesting geometrical reading in which the quantum potential emerges as active information determined by the vector of the superposition of Boltzmann entropies

$$\begin{cases} S_1 = k \log W_1(\theta_1, \theta_2, ..., \theta_p) \\ S_2 = k \log W_2(\theta_1, \theta_2, ..., \theta_p) \\ ... \\ S_n = k \log W_n(\theta_1, \theta_2, ..., \theta_p) \end{cases} \tag{11}$$

where W are the number of the microstates for the same parameters θ as temperatures, pressures, etc. . . .. In this picture, quantum effects are equivalent to a geometry which is described by the following equation

$$\frac{\partial}{\partial x^k} + \frac{\partial^2 S_j}{\partial x^k \partial x^p} \frac{\partial x^i}{\partial S_j} = \frac{\partial}{\partial x^k} + \frac{\partial \log W_j}{\partial x_h} = \frac{\partial}{\partial x^k} + B_h \tag{12}$$

where B_h is a Weyl-like gauge potential [19, 20]. In this picture, we have a deformation of the moments for the change of the geometry stated by the following expression of the

action:

$$\begin{aligned} A &= \int \rho[\tfrac{\partial A}{\partial t} + \tfrac{1}{2m}(p_i + B_i)(p_j + B_j) + V]dtd^n x \\ A &= \int \rho[\tfrac{\partial A}{\partial t} + \tfrac{1}{2m}(p_i p_j + B_i B_j) + V]dtd^n x \\ A &= \int \rho[\tfrac{\partial A}{\partial t} + \tfrac{1}{2m}(p_i p_j + \tfrac{\partial \log W}{\partial x_i}\tfrac{\partial \log W}{\partial x_j}) + V]dtd^n x \\ &= \int \rho[\tfrac{\partial A}{\partial t} + \tfrac{1}{2m}p_i p_j + V)dtd^n x + \tfrac{1}{2m}\tfrac{\partial \log W}{\partial x_i}\tfrac{\partial \log W}{\partial x_j})]dtd^n x \end{aligned} \quad . \tag{13}$$

The quantum action assumes the minimum value when

$$\delta A = 0$$

$$For$$

$$\delta \int \rho[\tfrac{\partial A}{\partial t} + \tfrac{1}{2m}p_i p_j + V)dtd^n x + \delta \tfrac{1}{2m}\tfrac{\partial \log W}{\partial x_i}\tfrac{\partial \log W}{\partial x_j})]dtd^n x = 0 \tag{14}$$

$$so$$

$$\tfrac{\partial A}{\partial t} + \tfrac{1}{2m}p_i p_j + V + \tfrac{1}{2m}(\tfrac{1}{W^2}\tfrac{\partial W}{\partial x_i}\tfrac{\partial W}{\partial x_j} - \tfrac{2}{W}\tfrac{\partial^2 W}{\partial x_i \partial x_j}) = \tfrac{\partial S}{\partial t} + \tfrac{1}{2m}p_i p_j + V + Q$$

where Q is the Bohm quantum potential that is a consequence for the minimum condition of Fisher information [21]. On the basis of equation (14), one can interpret Bohm's quantum potential as an information channel determined by the functions W defining the number of microstates of the physical system under consideration, which depend on the parameters θ of the distribution probability (and thus, for example, on the space-temporal distribution of an ensemble of particles, namely the density of particles in the element of volume d^3x around a point $\vec{x}$ at time t) and which correspond to the vector of the superpose Boltzmann entropies (11). In other words, the distribution probability of the wave function is linked to the functions W defining the number of microstates of the physical system. The quantum entropy emerges from these functions W given by equations (11), and can be considered as the fundamental physical entities which determine the action of the quantum potential (in the extreme condition of the Fisher information) on the basis of equation

$$Q = \frac{1}{2m}\left(\frac{1}{W^2}\frac{\partial W}{\partial x_i}\frac{\partial W}{\partial x_j} - \frac{2}{W}\frac{\partial^2 W}{\partial x_i \partial x_j}\right). \tag{15}$$

Under the constraint of the minimum condition of Fisher information, each "path" of the quantum potential is connected to the entropy by the functions W. In other words, each of the entropies appearing in the superposition vector (11) can be considered as a specific information channel of the quantum potential.

Moreover, on the basis of equations (12) and (14), one can say that the change of the geometry in the presence of quantum effects, which is expressed by a Weyl-like gauge potential is determined by the functions W and thus by the quantum entropy. Therefore, in non relativistic bohmian quantum mechanics the distribution probability of the wave function determines the functions W, the number of microstates of the system into consideration. The quantum entropy emerges from these functions W given by equations

(11), and determine a change of the geometry expressed by a Weyl-like gauge potential and characterized by a deformation of the moments given by equation (13).

Now, by introducing the definition (15) of the quantum potential as an geometric informational entity inside the quantum Hamilton-Jacobi equation, we obtain

$$\frac{|\nabla S|^2}{2m} + V + \frac{1}{2m}\left(\frac{1}{W^2}\frac{\partial W}{\partial x_i}\frac{\partial W}{\partial x_j} - \frac{2}{W}\frac{\partial^2 W}{\partial x_i \partial x_j}\right) = -\frac{\partial S}{\partial t} \tag{16}$$

which provides a new way to read the energy conservation law in quantum mechanics. In equation (16) two quantum corrector terms appear in the energy of the system, which are owed to the functions W linked with the quantum entropy, and which thus describe the change of the geometry in the presence of quantum effects. These two quantum corrector terms can thus be interpreted as a sort of degree of chaos of the background space determined by the ensemble of particles associated with the wave function under consideration. This opens the QM to the QFT and seems to suggest that the latter is the only authentic "realistic interpretation of QM [22]. It is also interesting to observe that the inverse square root of the quantity

$$L_{quantum} = \frac{1}{\sqrt{\frac{1}{\hbar^2}\left(\frac{2}{W}\frac{\partial^2 W}{\partial x_i \partial x_j} - \frac{1}{W^2}\frac{\partial W}{\partial x_i}\frac{\partial W}{\partial x_j}\right)}} \tag{17}$$

defines a typical quantum-entropic length that can be used to evaluate the strength of quantum effects and, therefore, the modification of the geometry with respect to the Euclidean geometry characteristic of classical physics. Once the quantum-entropic length becomes non-negligible the system goes into a quantum regime. In this picture, Heisenberg's uncertainty principle derives from the fact that we are unable to perform a classical measurement to distances smaller than the quantum-entropic length. So, the size of a measurement has to be bigger than the quantum-entropic length:

$$\Delta L \geq L_{quantum} = \frac{1}{\sqrt{\frac{1}{\hbar^2}\left(\frac{2}{W}\frac{\partial^2 W}{\partial x_i \partial x_j} - \frac{1}{W^2}\frac{\partial W}{\partial x_i}\frac{\partial W}{\partial x_j}\right)}}. \tag{18}$$

The quantum regime is entered when the quantum-entropic length must be taken under consideration.

Novello, Salim and Falciano [20] have recently proposed a geometrical approach in which the presence of quantum effects is linked with the Weyl length $L_W = \frac{1}{\sqrt{R}}$, and thus with the curvature scalar; in analogous way, in the approach here proposed, the quantum effects are owed to the microstates characterizing the system under consideration and thus the vector of the superposition of different Boltzmann entropies (11) emerges thus as the most fundamental source of quantum information.

5. The Geometry of Quantum Mechanics Interference

As Feynman stated, the double-slit interference "...has in it the heart of quantum mechanics. In reality, it contains the only mystery" [23]. It is thus the appropriate starting-

point to apply the entropic approach to quantum potential developed in chapter 4 and to illustrate the concept of quantum information as a measure of the deformation of the quantum entropy space.

Let us consider a wave function characterized by N probability densities $h_1, h_2, ..., h_n$:

$$|\psi\rangle = |h_1\rangle + |h_2\rangle + ... + |h_n\rangle \tag{19}$$

where $h_1 = \alpha_1 + i\beta_1$, $h_2 = \alpha_2 + i\beta_2$, $\ldots$, $h_n = \alpha_n + i\beta_n$.

The probability for the interference is

$$P(x) = \langle\psi \mid \psi\rangle = \sum_{i,j} g_{ij}\xi^i(x)\,\xi^j(x) \tag{20}$$

where

$$\xi^i = \sqrt{I^i}, \quad \xi^j = \sqrt{I^j} \tag{21}$$

and

$$g = \begin{bmatrix} 1 & \cos(\alpha_1 - \alpha_2) & ... & \cos(\alpha_1 - \alpha_2) \\ \cos(\alpha_2 - \alpha_1) & 1 & \cos(\alpha_2 - \alpha_1) & \cos(\alpha_2 - \alpha_1) \\ ... & ... & ... & ... \\ \cos(\alpha_n - \alpha_1) & \cos(\alpha_n - \alpha_2) & ... & 1 \end{bmatrix}. \tag{22}$$

On the basis of the metric (22), the square-distance between the end-points of two vectors ξ^i and η^i is

$$s^2(\theta_k) = \sum_{i,j} g_{i,j}\left(\xi^i(\theta_k) - \eta^i(\theta_k)\right)\left(\xi^j(\theta_k) - \eta^j(\theta_k)\right). \tag{23}$$

Moreover, when $\eta^i = \xi^i + \frac{\partial\xi^i}{\partial\theta_k}$ we obtain

$$ds^2 = \sum_{i,j} g_{i,j}\left(\frac{\partial\xi^i}{\partial\theta_k}\partial\theta_k\right)\left(\frac{\partial\xi^i}{\partial\theta_h}\partial\theta_h\right) = G_{h,k}\partial\theta^h\partial\theta^k \tag{24}$$

where

$$G = A^T gA \tag{25}$$

and

$$A = \begin{bmatrix} \frac{\partial\xi^1}{\partial\theta_1} & \frac{\partial\xi^1}{\partial\theta_2} & \cdots & \frac{\partial\xi^1}{\partial\theta_q} \\ \frac{\partial\xi^2}{\partial\theta_1} & \frac{\partial\xi^2}{\partial\theta_2} & \cdots & \frac{\partial\xi^2}{\partial\theta_q} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial\xi^n}{\partial\theta_1} & \frac{\partial\xi^n}{\partial\theta_2} & \cdots & \frac{\partial\xi^n}{\partial\theta_q} \end{bmatrix}. \tag{26}$$

On the basis of equations (24), (25) and (26), we have n quantum states and n parameters of the states in order to describe the process of interference [11].

Now, in the extreme condition of Fisher information given by equation (14), we can introduce the quantum-entropic length given by equation (17) inside equation (24) and thus we obtain:

$$ds^2 = G_{h,k}\partial\theta^h\partial\theta^k = \frac{1}{\frac{1}{\hbar^2}\left(\frac{2}{W}\frac{\partial^2 W}{\partial x_i \partial x_j} - \frac{1}{W^2}\frac{\partial W}{\partial x_i}\frac{\partial W}{\partial x_j}\right)}. \tag{27}$$

Equation (27) indicates clearly that the parameter states characterizing the quantum states in the process of the interference are determined by the functions W and thus by the quantum entropies. Incidentally, the (27) also suggests that in Physics the concept of distance always means a distinguishability between physical states. The probability distribution associated with the wave function it is fixed by the functions W defining the number of microstates of the beams of electrons. The interference process is associated with the quantum entropic length (27) and thus is determined by the quantum entropy. The vector of the superposed quantum entropies can be considered thus the ultimate physical reason in the experiment of the double-slit interference.

6. The Aharonov-Bohm Quantum Geometry

The second process we consider is the Aharonov-Bohm setup where a charged particle confined to a box acquires a geometric phase while slowly taking the box around a magnetic flux. The Aharonov-Bohm effect involves the quantum mechanical scattering of electrons in the presence of a classical magnetic vector potential produced by a current-carrying solenoid. Electrons are prevented from entering the region where the magnetic field itself is nonzero so there is no classical force on them. Nevertheless, information about the flux can be obtained.

As Aharonov and Bohm [24] showed, information about the magnetic flux (modulo a constant) through a solenoid can be measured via its effect on the interference pattern of electrons in a two-slit experiment. A particle of charge q traversing a path $C_\pm$ in the presence of a magnetic vector potential $\vec{A}$ can be described by a wave function $\psi_\pm(x)\, e^{\frac{iq}{\hbar c}\int_{C\pm}\vec{A}\cdot d\vec{l}}$ where $\psi_\pm(x)$ is the wave function in the absence of a vector potential and the integral is taken along the corresponding path $C_\pm$. We define the flux parameter φ as

$$\varphi = \frac{q}{\hbar c}\left(\int\limits_{C_+}\vec{A}\cdot d\vec{l} - \int\limits_{C_-}\vec{A}\cdot d\vec{l}\right) = \frac{q}{\hbar c}\oint\limits_{C}\vec{A}\cdot d\vec{l} = \frac{q}{\hbar c}\Phi \tag{28}$$

where C is a closed curve created by following C+ and returning along C- and Φ is the flux through the solenoid, and thus through any closed surface bounded by C. In terms of these variables, the total wavefunction at the screen is

$$\psi(x,\varphi) = e^{\frac{iq}{\hbar c}\int_{C_+}\vec{A}\cdot d\vec{l}}\left(\psi_+(x) + \psi_-(x)\, e^{-i\varphi}\right) \tag{29}$$

which is gauge invariant up to an overall phase.

To be more explicit, let us consider, for example, two wave packets A and B of an electron that passes on the two sides of a solenoid containing a magnetic flux Φ. On the basis of equation (5), when the line AB passes the solenoid, the wave packet A acquires a phase difference $\alpha = \frac{e}{\hbar c}\Phi$ with respect to the wave packet B. As a consequence, the expectation value of the modular momentum becomes $\exp\left(i\frac{e}{\hbar c}\Phi\right)/2$ and this change in the expectation value of the modular momentum by the factor $\exp\left(i\frac{e}{\hbar c}\Phi\right)/2$ as the line AB joining the two wave packets crosses the solenoid is the same in every gauge.

In the recent article *Quantum Measurement and the Aharonov-Bohm Effect* with Superposed Magnetic Fluxes, Bradonijc and Swain have considered what happens to the Aharonov-Bohm effect if the flux is in superposition of two states with different classical values [25]. These two authors have found that the interference pattern in the Aharonov-Bohm effect contains information about the nature of the superposition, allowing information about the state of the flux to be extracted without disturbing it. Bradonijc's and Swain's put in evidence that the information is obtained by a non-local operation Involving the vector potential without transfer of energy or momentum.By assuming that the current in the solenoid is in a superposition of two macroscopic states corresponding to equal and opposite currents, resulting in a superposition of positive and negative magnetic flux inside the solenoid, since the probability amplitude for a particle to be found at position x on the screen flux up due to vector potential $\vec{A}_{\uparrow}$ is

$$\psi_{\uparrow}(x,\varphi) = e^{\frac{iq}{\hbar c}\int\limits_{C_+}\vec{A}_{\uparrow}\cdot d\vec{l}}\left(\psi_+(x) + \psi_-(x)\,e^{i|\varphi|}\right) \tag{30}$$

and for the same flux down due to a vector potential $\vec{A}_{\downarrow}$ is

$$\psi_{\downarrow}(x,\varphi) = e^{\frac{iq}{\hbar c}\int\limits_{C_+}\vec{A}_{\downarrow}\cdot d\vec{l}}\left(\psi_+(x) + \psi_-(x)\,e^{-i|\varphi|}\right) \tag{31}$$

the general wave function of the particle is

$$|\psi(x,\varphi)\rangle = Cos\frac{\varphi}{2}\,|\psi_{\uparrow}\rangle + Sin\frac{\varphi}{2}e^{i\omega}\,|\psi_{\downarrow}\rangle \tag{32}$$

where $0 \leq \vartheta \leq \pi$ and $0 \leq \omega \leq 2\pi$. On the basis of the wave function (32), the total probability density at the screen is

$$|\psi(x,\varphi)|^2 = |\psi_+|^2 + |\psi_-|^2 + 2R\left(\psi_+^*\psi_-\right)\cos|\varphi| - 2J\left(\psi_+^*\psi_-\right)\sin|\varphi|\left(\cos^2\left(\frac{\vartheta}{2}\right) - \sin^2\left(\frac{\vartheta}{2}\right)\right). \tag{33}$$

It is interesting to remark that Bradonijc's and Swain's approach can be considered "as a toy model for the quantum mechanical propagation of a particle in a background spacetime which is a superposition of different classical geometries"[25].

Now, the quantum mechanical propagation of a particle in a background which is a superposition of different classical geometries, can be indeed seen, inside the geometrodynamics approach of the quantum potential in entropy space here suggested, as a consequence of the most general state of the background space given by the superposition of different Boltzmann entropies. The vector of the superposed entropies (11) can be indeed

seen as the fundamental entity which determines the quantum mechanical propagation of a particle in a background, given by a superposition of different classical geometries, characteristic of the Aharonov-Bohm effect. This result can be shown by appropriately including the quantum potential inside Bradonijc's and Swain's approach of Aharonov-Bohm effect.

In the pilot wave theory, the Aharonov-Bohm effect is explained through the local but indirect action of the vector potential on the particle via the quantum force [26, 27]. The Hamilton-Jacobi equation and the force law for this problem are

$$\frac{\partial S}{\partial t}+\frac{|\nabla S|^2}{2m}+eA_0+Q=0 \tag{34}$$

$$m\frac{d\vec{v}}{dt}=-\nabla Q+\vec{F} \tag{35}$$

where $\vec{F}$ is the Lorentz force, and the physical momentum $m\vec{v}=\nabla S-\frac{e}{c}\vec{A}$ and Q are gauge invariant quantities. Even when $\vec{F}=0$ the quantum potential is modified by $\vec{A}$. The latter may therefore be expected to cause a redistribution of the trajectories in the two-slit experiment. In the Aharonov-Bohm effect, the quantum potential may be expressed as

$$Q\left(\vec{x},\Phi\right)=Q_0\left(\vec{x}\right)+f\left(\vec{x},\Phi\right) \tag{36}$$

where $Q_0\left(\vec{x}\right)$ is independent of the flux. The term $Q_0\left(\vec{x}\right)$ is just determined by the vector of the superposed entropies (11) and thus can be assimilated to equation (15). By introducing the quantum potential (35) in the picture analysed by Bradonijc and Swain, we rewrite the wave function of a particle to be found at position x on the screen flux up due to vector potential $\vec{A}_\uparrow$ (namely the equation (30) as

$$\psi_\uparrow\left(x,\varphi\right)=e^{\frac{iq}{\hbar c}\int\limits_{C_+}\left(\vec{A}_\uparrow+Q_0+f(\vec{x},\Phi)\right)\cdot d\vec{l}}\left(\psi_+\left(x\right)+\psi_-\left(x\right)e^{i|\varphi|}\right) \tag{37}$$

and the wave function for the same flux down due to a vector potential $\vec{A}_\downarrow$ (equation (31)) as

$$\psi_\downarrow\left(x,\varphi\right)=e^{\frac{iq}{\hbar c}\int\limits_{C_+}\left(\vec{A}_\downarrow+Q_0+f(\vec{x},\Phi)\right)\cdot d\vec{l}}\left(\psi_+\left(x\right)+\psi_-\left(x\right)e^{-i|\varphi|}\right) \tag{38}$$

and thus the general wave function of the particle as

$$\psi_\uparrow\left(x,\varphi\right)=e^{\frac{iq}{\hbar c}\int\limits_{C_+}\left(\vec{A}_\uparrow+Q_0+f(\vec{x},\Phi)\right)\cdot d\vec{l}}\left(\psi_+\left(x\right)+\psi_-\left(x\right)e^{i|\varphi|}\right) \tag{39}$$

In equations (37), (38) and (39), the origin of the quantum potential $Q_0\left(\vec{x}\right)$ is just the vector of the superposed entropies (11), on the basis of the general definition of the quantum potential given by equation (15). Equations (37), (38) and (39) suggest that if the flux is in superposition of two states with different classical values, and thus the background is a superposition of different classical geometries, the quantum potential is modified by the vector potential in the sense that it contains, besides to the term (15) associated with the vector of the superposed entropies, a term depending itself on the

flux. The wave functions (37), (38) and (39) indicate that the change of the geometry of physical space determined by the Aharonov-Bohm effect is expressed by the quantities

$$A_{\uparrow} + \frac{1}{2m}\left(\frac{1}{W^2}\frac{\partial W}{\partial x_i}\frac{\partial W}{\partial x_j} - \frac{2}{W}\frac{\partial^2 W}{\partial x_i \partial x_j}\right) + f\left(\vec{x}, \Phi\right) \tag{40}$$

and

$$A_{\downarrow} + \frac{1}{2m}\left(\frac{1}{W^2}\frac{\partial W}{\partial x_i}\frac{\partial W}{\partial x_j} - \frac{2}{W}\frac{\partial^2 W}{\partial x_i \partial x_j}\right) + f\left(\vec{x}, \Phi\right). \tag{41}$$

On the basis of the quantities (40) and (41) describing the deformation of the geometry of physical space in the presence of the Aharonov-Bohm effect, the total probability density at the screen, given by equation

$$\left|\psi\left(x,\varphi\right)\right|^2 = \left|\psi_+\right|^2 + \left|\psi_-\right|^2 + 2R\left(\psi_+^*\psi_-\right)\cos\left|\varphi\right| - 2J\left(\psi_+^*\psi_-\right)\sin\left|\varphi\right|\left(\cos^2\left(\frac{\vartheta}{2}\right) - \sin^2\left(\frac{\vartheta}{2}\right)\right) \tag{42}$$

where ψ_+ and ψ_- are derived from equations (37) and (38), turns out to be determined just by these quantities (40) and (41).

Let us analyse now some interesting results which derive from equation (42) for various choices of ϑ.

1. For $\vartheta = 0$, $\left|\psi\left(x,\varphi\right)\right|^2$ goes to regular Aharonov-Bohm effect with an "up" flux.
2. For $\vartheta = \pi$, $\left|\psi\left(x,\varphi\right)\right|^2$ goes to regular Aharonov-Bohm effect with a "down" flux.
3. For $\vartheta = \pi/2$, the last term in equation (42) goes to zero. There is still an interference pattern, but it is different from the interference patterns in cases 1 and 2 above.

The first two limits indicate that that equation (42) reduces to the expected expressions for two classical flux states. More interesting is the third limit, which indicates that it is in principle possible to extract information about the quantum mechanical state of magnetic flux in the Aharonov-Bohm experiment from the electron diffraction pattern without disturbing the state of the flux. This information is extracted via a fundamentally nonlocal operation involving the deformation of physical space determined by the quantum potential and the vector potential (associated with the quantities (40) and (41)) and over an extended region of spacetime, and without any interaction in the region where the (superposition of) classical magnetic fields is present (i.e. the excluded region inside the solenoid). In other words, according to the geometry of quantum entropy space here suggested, the deformation of the physical space expressed by the quantities (40) and (41) can be seen as the source of information in the Aharonov-Bohm effect.

Moreover, in analogy with what we have seen in chapters 4 and 5 as regards the concept of a quantum-entropic length (equations (17) and (27) respectively), also as regards the Aharonov-Bohm effect, in the extreme condition of Fisher information given by equation (14), we can introduce a quantum-entropic length characterizing this quantum phenomenon given by the following equations:

$$L_{Aharonov-Bohm} = \frac{1}{\sqrt{\frac{1}{\hbar^2}\left(\frac{2}{W}\frac{\partial^2 W}{\partial x_i \partial x_j} - \frac{1}{W^2}\frac{\partial W}{\partial x_i}\frac{\partial W}{\partial x_j}\right) - 2m\left(A_{\uparrow} + f\left(\vec{x},\Phi\right)\right)}} \tag{43}$$

(which is the Aharonov-Bohm length associated with the region of physical space generating a flux up on the screen), and

$$L_{Aharonov-Bohm} = \frac{1}{\sqrt{\frac{1}{\hbar^2}\left(\frac{2}{W}\frac{\partial^2 W}{\partial x_i \partial x_j} - \frac{1}{W^2}\frac{\partial W}{\partial x_i}\frac{\partial W}{\partial x_j}\right) - 2m\left(A_{\downarrow} + f\left(\vec{x}, \Phi\right)\right)}} \tag{44}$$

(which is the Aharonov-Bohm length associated with the region of physical space generating a flux down on the screen). Equations (43) and (44) indicate clearly that the features of the background space in the presence of the Aharonov-Bohm effect are determined by the functions W (and thus by the quantum entropies), by the vector potential and the function linked with the phase difference. The Aharonov-Bohm effect is associated with the quantum entropic lengths (43) and (44) and thus is determined by the quantum entropy, the vector potential and the phase difference.

Although, in the Aharonov-Bohm setup, information is obtainable from the interference pattern associated with the deformation of physical space associated with the quantities (40) and (41), it is also important to mention that this information is not complete. In the classical Aharonov-Bohm effect, the flux Φ can only be determined modulo $2\pi\frac{\hbar c}{e}$. For $\Phi = 0$ modulo $2\pi\frac{\hbar c}{e}$, the interference pattern is the same as that for a positive and negative classical flux: there is no effect on the pattern. Superpositions of two magnetic fluxes (determined by the quantities (40) and (41) and thus associated with the geometries expressed by the quantum-entropic lengths (43) and (44)) which would individually be detectable via the Aharonov-Bohm effect can give rise to interference patterns which differ from any found in the classical (or non superposed) case.

It is also interesting to remark that the information about superposition is gathered from the full interference pattern. For any finite experimental resolution and finite number of electrons scattered, one can only construct the likelihood that the observed interference pattern corresponds to the prediction for an arbitrary superposition of fluxes. No single electron scattering event provides unambiguous information even about the flux modulo $2\pi\frac{\hbar c}{e}$.

One can also consider the case of fluxes which are detectable by the usual Aharonov-Bohm effect, but with unequal superpositions of "up" and "down" fluxes ($\left|\cos\frac{\vartheta}{2}\right|^2 \neq \left|\sin\frac{\vartheta}{2}\right|^2$). In this case, again, one extracts information about the nature of the state without any interaction which should cause the "collapse" of the state into one of definite flux.

To conclude our geometrodynamics treatment of the Aharonov-Bohm effect in the entropy space, it is finally important to mention the important role of the Berry phase. In his article A new phase in quantum computation Sjökvist has remarked that a geometric quantum information can be obtained that employs one-dimensional geometric phase factors. Sjökvist has written: "*The Berry phase, which occurs in situations like the Aharonov-Bohm setup [...] arises in adiabatic evolution, but now for non degenerate eigenspaces of Hamiltonians. Berry phases may be used for quantum computation by encoding the logical states in non degenerate energy levels, such as in the spin-up and spin-down states of a spin -1/2 particle in a magnetic field. When this field rotates*

slowly around a loop, the spin states will pick up Berry phases of magnitude given by half the enclosed solid angle and of opposite sign, which defines an adiabatic geometric phase-shift gate acting on the two spin states" [28]. The Berry phase in the Aharonov-Bohm effect can be just seen as a consequence of the vector of the superposed entropies, which are indeed the ultimate sources of quantum information, more precisely of the quantities (40) and (41) describing the deformation of the geometry in this process.

Conclusions

In this article a geometric approach to quantum information based on the quantum potential has been proposed. In this approach quantum information can be interpreted as a measure of the deformation of the physical background space determined by the quantum entropy and is associated with a quantum-entropic length. This approach provides a new suggestive geometric perusal of the double-slit interference and of the Aharonov-Bohm effect. It is increasingly recognized the role of emerging properties and global geometry from the collective behavior of qubit states. Apart from the mathematical interest, we believe that a geometric approach to quantum information can complement existing ones and release all the real possibilities of quantum computing [29, 30].

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Extending Differential Calculus for Noncommuting Variables

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Abstract: Recent innovations in the differential calculus for functions of non-commuting variables, begun for a quaternionic variable, are readily extended to the case of a general Clifford algebra and then extended more to the case of a variable that is a general square matrix over the complex numbers. The expansion of F(x+delta) is given to first order in delta for general matrix variables x and delta that do not commute with each other. Further extension leads to the broader study of commutators with functions, [y, f(x)] , expressed in terms of [y, x].

Keywords: Mathematical Physics; Differential Calculus; Noncommuting Variables; General Clifford Algebra
PACS (2010): 02.70.-c; 02.20.-a; 02.30.-f; 02.90.+p; 45.10.-b

1. Introduction

In a recent paper [1] I showed how to expand

$$F(x+\delta) = F(x) + F^{(1)}(x) + O(\delta^2) \tag{1..1}$$

when both x and δ were general quaternionic variables, thus did not commute with each other. It starts with the separation of the infinitesimal displacement into two parts, $\delta = \delta_{\|} + \delta_{\perp}$ as follows.

$$\delta_{\|} = \frac{1}{2}(1 - u_x \delta u_x), \qquad \delta_{\perp} = \frac{1}{2}(1 + u_x \delta u_x), \tag{1..2}$$

$$x = x_0 + ix_1 + jx_2 + kx_3 = x_0 + ru_x, \qquad r = \sqrt{x_1^2 + x_2^2 + x_3^2}. \tag{1..3}$$

This construction leads to the simple operator identities,

$$\delta_{\|}\, x = x\, \delta_{\|}, \qquad \delta_{\perp}\, x = x^*\, \delta_{\perp}, \tag{1..4}$$

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where x^* is the complex conjugate of x, having all imaginary components changed in sign. The final result is,

$$F^{(1)}(x) = F'(x)\delta_{\parallel} + [F(x) - F(x^*)](x - x^*)^{-1}\,\delta_{\perp}, \tag{1..5}$$

where $F'(x)$ is the ordinary derivative of the function $F(x)$.

It was noted in [1] that this result for quaternionic variables can be written in terms of a formal set of commutators:

$$F(x+\delta) = F(x) + F'(x)\delta_{\parallel} + [C, F(x)] + O(\delta^2), \qquad [C, x] = \delta_{\perp}. \tag{1..6}$$

It is a trivial matter to extend this result for quaternions to any Clifford algebra.

$$x = x_0 + \sum_{i=1,n} e_i\, x_i, \qquad e_i e_j + e_j e_i = -2\delta_{ij}, \tag{1..7}$$

since all we need to do is extend the definitions of u_x and r:

$$\sum_{i=1,n} e_i\, x_i = r u_x, \qquad r = \sqrt{\sum_{i=1,n} x_i^2} \tag{1..8}$$

and then use the very same formulas as in Eqs. (1..2), (1..5). There is some further material in Appendix D, where the venerable Fueter differential equation (for functions of a quaternionic variable) is extended to a general Clifford algebra.

The purpose of this paper is to extend that earlier analysis of differential calculus to a larger family of non-commuting variables. First we shall consider NxN matrices; then we shall look at more general formulations.

2. Finite Matrices as the Variable

Consider the N x N matrices X over the complex numbers and arbitrary analytic functions $F(X)$ with such a matrix as its variable. We seek a general construction for the first order term $F^{(1)}(X)$ when we expand $F(X+\Delta)$ given that Δ is small but still a general N x N matrix that does not commute with X.

The first step, following earlier work, is to represent the function F as a Fourier transform,

$$F(X) = \int dp f(p) e^{ipX} \tag{2..1}$$

where the integral may go along any specified contour in the complex p-plane; and then we also make use of the expansion,

$$e^{(X+\Delta)} = e^X[1 + \int_0^1 ds e^{-sX}\Delta\, e^{sX} + O(\Delta^2)]. \tag{2..2}$$

Another well-known expansion, relevant to what we see in (2..2), is

$$e^A\, B\, e^{-A} = B + [A, B] + \frac{1}{2!}[A, [A, B]] + \frac{1}{3!}[A, [A, [A, B]]] + \dots, \tag{2..3}$$

involving repeated use of the commutators, $[A, B] = AB - BA$.

The next step is to assume that we can find a matrix S that will diagonalize the matrix X at any given point in the space of such matrices.

$$A = S\ X\ S^{-1}, \qquad A_{i,j} = \delta_{i,j}\ \lambda_i, \qquad i, j = 1, \ldots, N \tag{2..4}$$

and we carry out the same transformation on the matrix Δ:

$$B = S\ \Delta\ S^{-1} \tag{2..5}$$

but, of course, the matrix B will not be diagonal.

Our task is to separate the matrix B into various parts, each of which will behave simply in the expansion of Eq. (2..3). The first step is to recognize that the diagonal part of the matrix B, call it B_0, commutes with the diagonal matrix A and thus we have

$$e^{tA}\ B_0\ e^{-tA} = B_0, \tag{2..6}$$

where we have $t = -isp$ from (2..1), (2..2).

Next we look separately at each off-diagonal element of the matrix B: that is, $B_{(i,j)}$ is the matrix that has only that one off-diagonal ($i \neq j$) element of the given matrix B, and all the rest are zeros. The first commutator is simply

$$[A, B_{(i,j)}] = r_{ij}\ B_{(i,j)}, \qquad r_{ij} = \lambda_i - \lambda_j \tag{2..7}$$

and then the whole series can be summed:

$$e^{tA}\ B_{(i,j)} e^{-tA} = e^{tr_{ij}}\ B_{(i,j)}. \tag{2..8}$$

Putting this all together, we have

$$S\ F^{(1)}\ S^{-1} = \int dp f(p)\ ip\ e^{ipA} \int_0^1 ds\ [B_0 + \sum_{i \neq j} e^{-ispr_{ij}}\ B_{(i,j)}]. \tag{2..9}$$

It is trivial to carry out the integrals over s; and we thus come to the final answer

$$F^{(1)}(X) = F'(X)\ \Delta_0 + \sum_{i \neq j} [F(X) - F(X - r_{ij} I)]\ r_{ij}^{-1}\ \Delta_{(i,j)} \tag{2..10}$$

where

$$\Delta_\mu \equiv S^{-1}\ B_\mu\ S, \qquad \mu = 0, (i, j). \tag{2..11}$$

The general structure of the result, Eq. (2..10), is similar to what we found in earlier work, Eq. (1..5): the first term $F'(X)$ looks like ordinary differential calculus and goes with that part of Δ that commutes with the local coordinate X; the remaining terms are non-local, involving the function F evaluated at discrete points separated from X by specific multiples of the unit matrix I.

While this final formula appears not as explicit as the previous result found for quaternionic variables, in any practical situation we have computer programs that can calculate the matrix operations referred to above with great efficiency.

A particular example, which was drawn out in [1], is the case of a variable built upon the matrices of any rank which generate the Lie algebra SU(2).

The quantities r_{ij}, which may be real or complex numbers, can be called the "roots" following the familiar treatment of Lie algebras. They have some properties, such as $r_{ij} = -r_{ji}$ and sum rules that involve traces of the original matrix X and powers of X.

What happens if the eigenvalues of X are degenerate? Suppose, for example, that $\lambda_1 = \lambda_2$. This means that r_{12} and r_{21} are zero. If we look at Eq. (2..10), we see that the terms $\Delta_{(1,2)}$ and $\Delta_{(2,1)}$ then have the coefficient $F'(x)$. Thus they simply add in with Δ_0. In the extreme case when all the eigenvalues are identical, then the answer is $F(X + \Delta) = F(X) + F'(X)\,\Delta + O(\Delta^2)$, which is old fashioned differential calculus for commuting variables.

Again, it is noted that the result Eq. (2..10) can be written as,

$$F(X + \Delta) = F(X) + F'(X)\Delta_0 + [C, F(X)] + O(\Delta^2), \qquad [C, X] = \Delta - \Delta_0. \tag{2..12}$$

3. General Commutator Formulation

Now we shall explore the more abstract form of the results reported above which involves commutator equations for any noncommutative system.

$$F^{(1)}(x) = F'(x)\,\delta_0 \; + [C, F(x)], \tag{3..1}$$

$$[C, x] = \delta - \delta_0, \qquad [\delta_0, x] = 0. \tag{3..2}$$

Here the operator C is defined implicitly, through its commutator with x, rather than explicitly; and δ_0 is that portion of δ that commutes with x (called $\delta_\|$ or Δ_0 above). One may readily confirm the correctness of this formulation in the case of $F(x) = x^n$.

Here is a more formal-looking derivation of these equations. [2] Start with the following identity for suitably smooth functions $F(x)$.

$$e^C F(x) e^{-C} = F(e^C x e^{-C}), \tag{3..3}$$

$$F(x) + [C, F(x)] + \ldots = F(x + [C, x] + \ldots), \tag{3..4}$$

where the dots indicate terms of second order or higher in the quantity C, which is assumed to be small. If we now define the operator C by $[C, x] = \delta - \delta_0$, with $[\delta_0, x] = 0$, this yields,

$$F(x + \delta - \delta_0) = F(x + \delta) - F'(x)\,\delta_0 + \ldots = F(x) + [C, F(x)] + \ldots. \tag{3..5}$$

which connects Eq. (3..1) to the original statement of the differential calculus in Eq. (1..1).

It is interesting that this new formalism manages to hide the non-locality, which was a prominent feature of the previous examples.

In the rest of this paper I want to explore this commutator formalism as a general mathematical problem: how to express a commutator with a function $[y, f(x)]$ in terms of $[y, x]$.

In any non-commutative (but associative) algebra, we have the basic identity,

$$[x, yz] = y[x, z] + [x, y]z; \tag{3..6}$$

and repetition of this identity leads to the familiar formula,

$$[y, x^n] = \sum_{m=0}^{n-1} x^m \, [y, x] \, x^{n-m-1}. \tag{3..7}$$

We want to seek similar expressions for some familiar functions of x:

$$[y, f(x)] = \textit{ something built around } [y, x]. \tag{3..8}$$

Some earlier studies [3] of this general problem assumed that the commutator $[y, x]$ is a constant; but here we want to go beyond that assumption.

4. Non-Integral Power

Let's start with $f(x) = x^\nu$ for an arbitrary number ν. We introduce a parameter p, which commutes with x and y, and consider the function:

$$Q_\nu(p) = [y, (x + p)^\nu]. \tag{4..1}$$

We have the algebraic identity

$$Q_\nu(p) = [y, (x + p)(x + p)^{\nu-1}] = (x + p)Q_{\nu-1}(p) + [y, x](x + p)^{\nu-1}; \tag{4..2}$$

and we can also take the derivative,

$$\frac{dQ_\nu(p)}{dp} = \nu \; Q_{\nu-1}(p). \tag{4..3}$$

Combining these two equations we get a simple differential equation for $Q_\nu(p)$, which is easily solved.

$$Q_\nu(p) = \nu \; (x + p)^\nu \int_p^\infty ds \; (x + s)^{-\nu-1} \; [y, x] \; (x + s)^{\nu-1}. \tag{4..4}$$

Now, setting $p = 0$, we have the desired result,

$$[y, x^\nu] = \nu \; x^\nu \int_0^\infty ds \; (x + s)^{-\nu-1} \; [y, x] \; (x + s)^{\nu-1}. \tag{4..5}$$

An alternative version of this formula, starting with a rewrite of Eq. (4..2), is

$$[y, x^\nu] = \nu \int_0^\infty ds \; (x + s)^{\nu-1} \; [y, x] \; (x + s)^{-\nu-1} \; x^\nu. \tag{4..6}$$

One special case of this general formula is the familiar result,

$$[y, x^{-1}] = -x^{-1} \; [y, x] \; x^{-1}; \tag{4..7}$$

and another, unfamiliar, special case comes from taking the limit as ν goes to zero:

$$[y, ln(x)] = \int_0^\infty ds \, \frac{1}{x+s} \, [y, x] \, \frac{1}{x+s}. \tag{4..8}$$

From this we can derive the following.

$$[y, tan^{-1}(x)] = \int_0^1 ds \, \frac{1}{1+s^2x^2} \Big([y, x] - s^2 x[y, x]x \Big) \frac{1}{1+s^2x^2}. \tag{4..9}$$

An alternative representation can be found, using the formula (A.1) in Appendix A, for the case $-1 < \nu < 0$:

$$[y, x^\nu] = \frac{sin \; \pi\nu}{\pi} \int_0^\infty ds \; s^\nu \, \frac{1}{x+s} \, [y, x] \, \frac{1}{x+s}. \tag{4..10}$$

This can also be written as,

$$[y, x^\nu] = \mu \frac{sin \; \pi\nu}{\pi} \int_0^\infty ds \, \frac{1}{x+s^\mu} \, [y, x] \, \frac{1}{x+s^\mu} \; , \qquad \mu = 1/(\nu+1). \tag{4..11}$$

5. Exponential

To study the exponential function, we again introduce a real parameter,

$$R(p) = [y, e^{px}]. \tag{5..1}$$

We take the derivative with respect to p,

$$\frac{dR(p)}{dp} = [y, xe^{px}] = xR(p) + [y, x]e^{px}, \tag{5..2}$$

and solve this simple differential equation to get the result:

$$[y, e^x] = \int_0^1 ds \; e^{(1-s)x} \, [y, x] \, e^{sx}. \tag{5..3}$$

I would not be surprised if this formula has been discovered before. A very similar looking formula, which is well-known, is

$$\frac{d}{d\lambda} \, e^{x(\lambda)} = \int_0^1 ds \; e^{(1-s)x} \, \frac{dx(\lambda)}{d\lambda} \, e^{sx}. \tag{5..4}$$

This result, Eq. (5..3), can also be derived from Eq. (4..5) if we make the substitution, $x \to (1 + x/\nu)$ and take the limit $\nu \to \infty$.

6. Some Other Familiar Functions

The Chebyshev polynonials of the second kind $U_n(x)$ are defined by a simple generating function.[4]

$$\frac{1}{1 - 2xt + t^2} = \sum_{n=0}^\infty t^n \, U_n(x). \tag{6..1}$$

If we take the commutator of this equation with y, we find

$$\sum_n t^n\, [y, U_n(x)] = 2t\frac{1}{1-2xt+t^2}\, [y,x]\, \frac{1}{1-2xt+t^2}. \qquad (6..2)$$

Now using the expansion (6..1) again; and collecting terms with the same power of t, we find,

$$[y, U_n(x)] = 2\sum_{m=0}^{n-1} U_m(x)\, [y,x]\, U_{n-m-1}(x). \qquad (6..3)$$

This is a remarkably simple formula; it looks similar to the earliest Eq. (3..7).

One can get a very similar-looking formula for the set of polynomials generated by $1/(1 - xt + \chi(t))$, where $\chi(t)$ is any power series starting off with t^2. The special case $\chi = 0$ gives us Eq. (3..7). An application of this sort of generating function, with multiple noncommuting variables, may be seen in an earlier study of infinite dimensional free algebra in large N matrix theory.[5]

The Chebyshev polynomials of the first kind $T_n(x)$ come from a slightly different generating function and they yield the following formula.

$$[y, T_n(x)] = -[y,x]\, U_{n-1}(x) + 2\sum_{m=0}^{n-1} T_m(x)[y,x]\, U_{n-m-1}(x). \qquad (6..4)$$

Gegenbauer polynomials are given by this generating function,

$$(1-2xt+t^2)^{-\nu} = \sum_{n=0}^{\infty} t^n\, C_n^{\nu}(x). \qquad (6..5)$$

For $\nu = 1$ these are the Chebyshev and for $\nu = 1/2$ these are the Legendre polynomials. If I try to use the formulas above directly, then it results in a very complicated mess. In Appendix A is a formula that relates this generating function Eq. (6..5) to an integral over the simpler one of Eq. (6..1),

$$C_n^{\nu}(x) = \frac{sin\pi\nu}{\pi}\, 2\int_0^1 du\, u^{n-1}\, (1/u^2-1)^{-\nu}\, U_n(ux), \qquad 0<\nu<1; \qquad (6..6)$$

and this lets us calculate $[y, C_n^{\nu}(x)]$ using results above.

The Laguerre polynomials are given by another generating function,

$$(1-t)^{-\alpha-1}e^{-xt/(1-t)} = \sum_{n=0}^{\infty} t^n\, L_n^{(\alpha)}(x). \qquad (6..7)$$

Following procedures given above, we find the commutator equation,

$$[y, L_n^{(\alpha)}(x)] = -\int_0^1 ds\sum_{m=0}^{n-1} L_m^{(\alpha)}((1-s)x)\, [y,x]\, L_{n-m-1}^{(0)}(sx), \qquad (6..8)$$

which involves an integral and a finite sum.

Bessel functions of integral order are given from this generating function:

$$e^{x(t-t^{-1})/2} = J_0(x) + \sum_{n=1}^{\infty}[t^n + (-t)^{-n}]J_n(x). \qquad (6..9)$$

If we take the commutator with y, use Eq. (5..3) and again compare the coefficients of each power of t, we get a rather complicated formula, involving an integral and an infinite sum. Here is the result for J_0.

$$[y, J_0(x)] = \int_0^1 ds \sum_{m=0}^{\infty} (-1)^{m+1}[J_m((1-s)x)\ [y,x]\ J_{m+1}(sx) + \tag{6..10}$$

$$J_{m+1}((1-s)x)\ [y,x]\ J_m(sx)]. \tag{6..11}$$

Another representation for some Bessel functions is the following [6].

$$x^\mu\ H_\mu^{(1)}(x) = \frac{2}{\pi i} \int_0^\infty ds\ s^{\mu+1}\ J_\mu(s)\ \frac{1}{s^2 - x^2}, \tag{6..12}$$

where the variable x lies in the first quadrant of the complex plane. Applying the commutator $[y, \ \ ]$ to this equation will give a simpler-looking formula.

For more in this vein, see Appendix B.

One more famous function is the Gamma function $\Gamma(x)$ and we find the following formula for its logarithmic derivative.

$$\psi(x) = \frac{1}{\Gamma(x)} \frac{d\Gamma(x)}{dx}, \qquad [y, \psi(x)] = \sum_{n=0}^{\infty} \frac{1}{n+x}[y,x]\frac{1}{n+x}, \tag{6..13}$$

which looks like the discrete analog of Eq. (4..8); and from this we also get,

$$[y, cot(x)] = -\sum_{n=-\infty}^{\infty} \frac{1}{n\pi + x}\ [y,x]\ \frac{1}{n\pi + x}. \tag{6..14}$$

7. Discussion

The results given in this paper are not only new, they are mostly of an unfamiliar form. The formulas in Sections 4 and 5 involve definite integrals (over a parameter s) that would be easy to evaluate if it were not for the operator $[y,x]$ that stands in the midst of simple functions of s and x. I suggest an interesting exercise for readers is to take the special case $y = d^2/dx^2$ and see how those integrals work out. But also see Appendix C.

Are there any restrictions on the formulas derived above? Within Section 4, I should specify that the operator x has eigenvalues with positive real parts. This is necessary to give meaning to the nonintegral power of x and it will avoid singularities in the integrals over the parameter s.

In Sections 4 and 5 there are formulas for the commutator of y with x^ν, $ln(x)$, and e^{px}; and one should be able to relate those formulas. For example,

$$[y, x^\nu] = [y, e^{\nu ln x}] = \int_0^1 ds\ e^{(1-s)\nu ln(x)}\ [y, \nu ln(x)]\ e^{s\nu ln(x)} = \tag{7..1}$$

$$\nu\ x^\nu \int_0^1 ds\ x^{-\nu s} \int_0^\infty dt\ \frac{1}{t+x}\ [y,x]\ \frac{1}{t+x}\ x^{\nu s}. \tag{7..2}$$

How do we get this to look like Eq. (4..5) ?

Appendix A: Relating Generators

We want to relate the generating function in Eq. (6..5) to that in Eq. (6..1). We start with a simple integral identity,

$$\int_0^\infty \frac{d\zeta}{\zeta + M}\,\zeta^{-\nu} = \frac{\pi}{\sin(\pi\nu)} M^{-\nu}, \qquad M > 0, \qquad 0 < \nu < 1. \tag{A.1}$$

For the problem at hand we have $M = 1 - 2xt + t^2$. Now we can write the generating function for the Gegenbauer polynomials as,

$$\sum_{n=0}^{\infty} t^n\, C_n^\nu(x) = \frac{sin(\pi\nu)}{\pi} \int_0^\infty d\zeta\, \zeta^{-\nu}\, \frac{1}{\zeta + 1 - 2xt + t^2}. \tag{A.2}$$

Now we just need to scale the variables t and x to get,

$$C_n^\nu(x) = \frac{sin(\pi\nu)}{\pi} \int_0^\infty d\zeta\, \zeta^{-\nu} \frac{1}{\zeta+1}\,\frac{1}{q^n}\, U_n(x/q), \qquad q = \sqrt{\zeta + 1}. \tag{A.3}$$

Appendix B: A More Formal Approach

Given any analytic function $f(x)$ we can write the equation,

$$f(x) = \frac{1}{2\pi i} \oint ds\, f(s)\, \frac{1}{s - x}. \tag{B.1}$$

where the countour of the integral in the complex s-plane encloses the point x. Then, taking the commutator, we have,

$$[y, f(x)] = \frac{1}{2\pi i} \oint ds\, f(s) \frac{1}{s - x}\, [y, x]\, \frac{1}{s - x}. \tag{B.2}$$

How one might make practical use this equation is unclear.

Appendix C: Troublemaking

To illustrate the unfamiliarity of the "integrals-with-operators" seen in this paper, lets start with the earliest result, Eq. (4..5).

$$[y, x^\nu] = \nu\, x^\nu \int_0^\infty ds\, (x + s)^{-\nu-1}\, [y, x]\, (x + s)^{\nu-1}. \tag{C.1}$$

Let me now scale the integration parameter as $s = xt$. This should be ok since we also noted that we would want x to have positive real values. Then this formula would read,

$$[y, x^\nu] = \nu x^\nu \int_0^\infty xdt\, x^{-\nu-1}\, (1 + t)^{-\nu-1}\, [y, x]\, x^{\nu-1}\, (1 + t)^{\nu-1}. \tag{C.2}$$

It would now appear that we can move the factors $(1+t)$ past the operators y and x and then simply evaluate the t-integral, leaving

$$[y, x^\nu] = \nu\, [y, x]\, x^{\nu-1}, \tag{C.3}$$

which is clearly wrong! The error may be ascribed to the step where we moved $(1+t)$ past the operator y, since this new variable t was constructed to involve x, which does not commute with y.

Appendix D: Fueter Equation for a General Clifford Variable

Fueter [7], long ago, studying calculus with a quaternionic variable, found a third order differential equation that any "holomorphic" function of such a variable would satisfy. We generalize that result for noncommuting variables based on a general Clifford algebra.

For reference, we start with the familiar Cauchy-Riemann equation for an analytic function of a complex variable,

$$(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y})f(z = x + iy) = 0. \tag{D.1}$$

Now we want to consider the non-commuting variables x that are based on an n-dimensional Clifford algebra,

$$x = x_0 + \sum_{i=1,n} e_i\, x_i, \qquad e_i e_j + e_j e_i = -2\delta_{i,j} \ \ , \tag{D.2}$$

and the quantities x_0, x_i are real numbers. The procedure we follow to get the generalized Fueter differential equation is that given in [1] for quaternions (n=3).

Define the first order differential operator,

$$\Box_{n+1} = \frac{\partial}{\partial x_0} + \sum_{i=1,n} e_i\, \frac{\partial}{\partial x_i}, \tag{D.3}$$

and see how that acts upon the particular function $F(x) = e^{px}$ for some real parameter p.

$$e^{px} = e^{px_0}(cos\ pr + \Sigma\ \frac{sin\ pr}{r}), \tag{D.4}$$

$$r = \sqrt{\sum_{i=1,n} x_i^2}, \qquad \Sigma = \sum_{i=1,n} e_i x_i, \qquad \Sigma^2 = -r^2 \tag{D.5}$$

$$\Box_{n+1}\, e^{px} = (1-n)\, e^{px_0}\frac{sin\ pr}{r}. \tag{D.6}$$

For $n = 1$ we have the Cauchy-Riemann formula, Eq. (D.1). For $n = 3$ we see that the right hand side of Eq. (D.6) vanishes under action of the Laplacian operator in 4-dimensions.

$$\triangle_4 = \Box_4^*\, \Box_4 = \frac{\partial^2}{\partial x_0^2} + \frac{1}{r}\frac{\partial^2}{\partial r^2}r. \tag{D.7}$$

This gives us Fueter's third order differential equation for functions of a quaternionic variable.

$$\triangle_4\, \Box_4\, F(x) = 0, \tag{D.8}$$

assuming that any function $F(x)$ can be written as a real superposition of the exponentials e^{px}.

Now, how do we go for $n > 3$? It appears that we can still write,

$$\text{“}\triangle_4\text{”} \ \Box_{n+1} \ F(x) = 0, \tag{D.9}$$

where I have put “quotation marks” around that operator $\triangle_4$ to mean that it is literally what is written in Eq. (D.7) and the quantity r is meant to be the n-dimensional radial distance defined in Eq. (D.5). So, this is not really the Laplacian operator in n+1 dimensions; but it is a third order differential equation that is satisfied.

Alternatively, it can be shown that one can introduce the correct Laplacian operator in n+1 dimensions,

$$\triangle_{n+1} = \Box^*_{n+1} \ \Box_{n+1} = \frac{\partial^2}{\partial x_0^2} + \frac{\partial^2}{\partial r^2} + \frac{n-1}{r}\frac{\partial}{\partial r}, \tag{D.10}$$

and find the following formula, an n-th order differential equation, which appears to be valid only for odd values of n.

$$(\triangle_{n+1})^{(n-1)/2} \ \Box_{n+1} \ F(x) = 0. \tag{D.11}$$

This last result has been found earlier by researchers [8] who followed the classical line of Fueter's analysis. But the earlier formula Eq. (D.9) appears to be new.

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Duality Theory of Strong Interaction

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Abstract: The main objective of this article is to explore the duality of strong interaction derived from an $SU(3)$ gauge theory based on two basic principles: the principle of interaction dynamics (PID) and principle of representation invariance (PRI). Intuitively, PID takes the variation of the action functional under energy-momentum conservation constraint, and PRI requires that physical laws be independent of representations of the gauge group. The new $SU(3)$ gauge field equations establish a natural duality between strong gauge fields $\{S^k_\mu\}$, representing the eight gluons, and eight bosonic scalar fields. With the duality, we derive three levels of strong interaction potentials: the quark potential Φ_q, the nucleon/hadron potential Φ_n and the atom/molecule potential Φ_a. These potentials clearly demonstrates many features of strong interaction consistent with observations. In particular, they provide a clear explanation for both quark confinement and asymptotic freedom. Also, in the nuclear level, the new potential is an improvement of the Yukawa potential. As the distance between two nucleons is increasing, the nuclear force corresponding to the nucleon potential Φ_n behaves as repelling, then attracting, then repelling again and diminishes, consistent with experimental observations.

Keywords: Strong Interaction, Duality Theory, Field Equations, QCD, Quark Potentials, Nucleon Potential, Atom Potential, Quark Confinement, Asymptotic Freedom, Short-Range Nature of Strong Interaction
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1. Introduction

Strong interaction, electromagnetic interaction, weak interaction and gravity are four fundamental interactions of Nature. The main objective of this article is to establish a duality theory for strong interaction based on an $SU(3)$ gauge theory, the principle of interaction dynamics (PID) and the principle of representation invariance (PRI), postulated recently by the authors [14, 15].

Intuitively, PID takes the Lagrangian action with energy-momentum conservation constraint. The original motivation for PID was to derive gravitational field equations taking into account of the presence of dark energy and dark matter [16]. The key point was that due to the presence of dark energy and dark matter, the energy-momentum tensor $\{T_{\mu\nu}\}$ is no longer conserved, and the variation of the classical Einstein-Hilbert functional has to be taken under the divergence-free constraint. Namely, the variational element must be energy-momentum conserved.

With PID at our disposal, we derive in [14, 15] a unified field model. This model leads not only to consistent results with the standard model, but also to many new insights and predictions. One important feature of the unified field model is a natural duality between the interacting fields (g, A, W^a, S^k), corresponding to graviton, photon, intermediate vector bosons $W^{\pm}$ and Z and gluons, and the dual bosonic fields $(\Phi_\mu, \phi^0, \phi^a_w, \phi^k_s)$.

The second principle PRI requires that all $SU(N)$ gauge theories should be invariant under transformations of different representations of the gauge group $SU(N)$. In a sense, this principle has the same spirit as the principle of general relativity, where physical laws should be independent of the coordinate system. For gauge theories in particle physics, it is equally natural to require physical laws be independent of different representations of the underlying Lie group $SU(N)$. It is with PRI that we substantially reduced the large number of to-be-determined parameters in the unified model to two $SU(2)$ and $SU(3)$ constant vectors $\{\alpha^1_\mu\}$ and $\{\alpha^2_k\}$, containing 11 parameters, which represent the portions distributed to the gauge potentials by the weak and strong charges.

The field equations can be decoupled easily to study individual interactions. In this article, the new gauge field equations for strong interaction are derived by applying PID to the standard $SU(3)$ gauge action functional in QCD. The new model leads to consistent results as the classical QCD, and, more importantly, to a number of new results and predictions.

First, this model gives rise to a natural duality between the $SU(3)$ gauge fields S^k_μ $(k = 1, \cdots, 8)$, representing the gluons, and the dual scalar fields ϕ^k_s.

Second, for the first time, we derive several levels of strong interaction potentials, including the quark potential S_q, the nucleon potential S_n and the atom/molecule potential

S_a. They are given as follows:

$$\Phi_q = g_s \left[\frac{1}{r} - \frac{A_q}{\rho}(1 + k_0 r)e^{-k_0 r} \right], \tag{1..1}$$

$$\Phi_n = 3g_s \left(\frac{\rho_0}{\rho_n} \right)^3 \left[\frac{1}{r} - \frac{A_n}{\rho_n}(1 + k_1 r)e^{-k_1 r} \right], \tag{1..2}$$

$$\Phi_a = 3N g_s \left(\frac{\rho_0}{\rho_a} \right)^3 \left[\frac{1}{r} - \frac{A_a}{\rho_a}(1 + k_1 r)e^{-k_1 r} \right], \tag{1..3}$$

where g_s is the strong charge[2], A_q, A_n and A_a are dimensionless constants, $r_0 = 1/k_0 = 10^{-16}$cm, $r_1 = 1/k_1 = 10^{-13}$cm, ρ_0 is the effective quark radius, ρ_n is the effective radius of a nucleon, ρ_a is the radius of an atom/molecule, and N is the number of nucleons in an atom/molecule. These potentials match very well with experimental data, and offer a number of physical conclusions. Hereafter we shall explore a few important implications of these potentials.

THIRD, with these potentials, the binding energy of quarks can be estimated as

$$E_q \sim \left(\frac{\rho_n}{\rho_0} \right)^6 E_n \sim 10^{24} E_n \sim 10^{22}\text{GeV}, \tag{1..4}$$

where $E_n \sim 10^{-2}$GeV is the binding energy of nucleons. Consequently, close to the Planck energy scale is required to break a quark free. Hence these potential formulas offer a clear mechanism for quark confinement.

FOURTH, with the quark potential, there is a radius $\bar{r}$, as shown in Figure 6..1, such that two quarks closer than $\bar{r}$ are repelling, and for r near $\bar{r}$, the strong interaction diminishes. Hence this clearly explains asymptotic freedom.

FIFTH, in the nucleon level, the new potential is an improvement of the Yukawa potential. The corresponding Yukawa force is always attractive. However, as the distance between two nucleons is increasing, the nucleon force corresponding to the nucleon potential S_n behaves as repelling, then attracting, then repelling again and diminishes. This is exactly the picture that the existing observations tell us.

SIXTH, the factor $\left(\frac{\rho_0}{\rho_2}\right)^3$ in (1..3) indicates the short-range nature of the strong interaction, in agreement with observations. In particular, beyond molecular level, strong interaction diminishes.

Before the end of this Introduction, we would like to give a brief historical account of the related problems studied in this article.

1. The use of gauge symmetry and the non-abelian gauge theory for particle physics go back to [12, 29], motivated also by the work of [28]. An important revolution regarding was [25], where the Yang-Mills theory was shown to be renormalizable, even after symmetry breaking.

[2] Here g_s is the strong charge using quark as a reference level; see Remark 4..1.

With the great achievements in field theory and particle physics in the last hundred years, it is clear that the principle of gauge symmetry is one of the few key principles for electromagnetism, weak and strong interactions. It is also one of few first principles we build the field theory in this and the accompanying articles.

2. The spontaneous symmetry breaking was another revolution in field theory and in particle physics, going back to [18, 20, 21, 1, 4], followed by the 1964 papers on the introduction to the Higgs field [2, 9, 7]. The success of the classical electroweak theory [26, 3, 24], together with the Gell-Mann's quark theory and Greenberg's introduction of color charge, leads to the standard model; see, among many others [10, 23, 5, 11, 8] and the references therein.

As mentioned earlier, our studies are based on two newly postulated principles, called principle of interaction dynamics (PID) and the principle of representation invariance (PRI). Again, PID takes the variation of the Lagrangian action subject to energy-momentum conservation constraints. This new principle gives an entirely different way of introducing Higgs fields, leading to a natural approach for spontaneous symmetry breaking and mass generation.

3. The introduction of PRI is based purely on mathematical logic, and its validity is unquestionable. Hence any physical theory violating PRI may only be considered as an approximation of laws of Nature.

A crucial consequence of PRI is that for the gauge fields A^a_μ and G^k_μ, corresponding to two different gauge groups $SU(N_1)$ and $SU(N_2)$, the following combinations

$$\alpha A^a_\mu + \beta G^k_\mu$$

are prohibited, since this combination violates PRI. This point of view clearly shows that both the classical electroweak theory and the standard model violates PRI. Of course, this violation does not in any way undermine the importance of the classical electroweak theory and the standard model — as the Einstein theories of relativity do not undermine the Newton laws of physics.

4. There are many attempts to physics beyond standard model. One such attempt is to derive a unified theory based on large gauge group such as $SU(5)$. Unfortunately, the apparent violation of PRI for such theories forces us to take a different route, which has the spirit similar to that of the standard model. Namely, a consistent unified field theory should obey the following requirements:

(1) Its Lagrangian action density must satisfy all known basic symmetry principles, including the principle of general relativity, the principle of Lorentz invariance, the principle of gauge invariance, and the principle of representation invariance (PRI);

(2) The Lagrangian action can be naturally decoupled to Lagrangian actions for individual interactions: the Einstein-Hilbert action for gravity, the $U(1)$ gauge field action for electromagnetism, the $SU(2)$ gauge field action for weak interaction and the $SU(3)$ gauge field action for strong interaction.

In fact, the unified field theory developed by the authors [14, 15] is uniquely determined by these two requirement.

Also, our unified field model appears to match Nambu's vision. In fact, in his Nobel lecture [19], Nambu stated that

> *Einstein used to express dissatisfaction with his famous equation of gravity*
>
> $$G_{\mu\nu} = 8\pi T_{\mu\nu}$$
>
> *His point was that, from an aesthetic point of view, the left hand side of the equation which describes the gravitational field is based on a beautiful geometrical principle, whereas the right hand side, which describes everything else, . . . looks arbitrary and ugly.*
>
> *... [today] Since gauge fields are based on a beautiful geometrical principle, one may shift them to the left hand side of Einsteins equation. What is left on the right are the matter fields which act as the source for the gauge fields ... Can one geometrize the matter fields and shift everything to the left?*

Our understanding of his statement is that the left-hand side of the standard model is based on the gauge symmetry principle, and the right-hand side of the standard model involving the Higgs field is artificial. What Nambu presented here is a general view shared by many physicists that the Nature obeys simple beautiful laws based on a few first principles.

5. In another accompanying article [17], we introduce a weakton model of elementary particles, leading to explanations for sub-atomic decays and the creation/annihilation of matter/antimatter particles, and for the baryon asymmetry problem. Remarkably, in the weakton model, both the spin-1 mediators (the photon, the W and Z vector bosons, and the gluons) and the spin-0 dual mediators introduced in the unified field model have the *same* weakton constituents, differing only by their spin arrangements. The spin arrangements clearly demonstrate that there must be dual mediators with spin-0. This observation clearly supports the unified field model presented in [14, 15]. Conversely, the existence of the dual mediators makes the weakton constituents perfectly fit.

6. Quantum chromodynamics (QCD) was established as an $SU(3)$ quantum gauge field theory for strong interaction; see among many others [10, 5, 23] for the historical accounts.

Three important properties of strong interaction are the quark confinement, asymptotic freedom–discovered in the early 1970s by David Politzer [22] and by Frank Wilczek and David Gross [6]– and the short-range nature. Of course, among many other features of QCD, they are many important studies regarding to these three properties of strong interaction; see the references listed at the end of Chapter 8 of [23].

However due to the complexity of strong interaction, one still does not know how to derive these properties directly from the QCD gauge field theory. One difficulty comes from the fact that there are eight $SU(3)$ gauge fields, representing eight gluons. Strong interaction is taken place under the combined influence of *at least* these eight gauge fields. Fortunately, with PRI, as we mentioned earlier, we are able to find such combined physical quantity S_μ defined by (3..3).

Also, from the derived strong interactions potentials (1..1)–(1..3) above, it is clear that the quark confinement is mainly due to the effect of the dual gluon fields, represented by ϕ^s_k on the right hand side of the $SU(3)$ gauge field equations (3..1).

The paper is organized as follows. Section 2 introduces two basic principles, PID and PRI. Sections 3 and 4 derive the strong interaction potentials. Section 5 explains the quark confinement and asymptotic freedom using the strong interaction potentials derived in Sections 3 and 4. Section explores the connections to the Yukawa potential and explains the short-range nature of the strong interaction.

2. Recapitulation of Two Basic Principles and Unified Field Theory

We introduce in this section the unified field theory based on two basic principles, principle of interaction dynamics (PID) and principle of representation invariance (PRI), postulated recently by the authors [14, 15].

2.1 Principle of Representation Invariance

The $SU(N)$ gauge theory provides an action

$$L = \int \mathcal{L} dx, \tag{2..1}$$

with the action density

$$\mathcal{L} = -\frac{1}{4} G_{ab} F^a_{\mu\nu} F^{b\mu\nu} + \bar{\Psi}(i\gamma^\mu D_\mu - m)\Psi, \tag{2..2}$$

where

$$\begin{aligned} &F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g\lambda^a_{bc} A^b_\mu A^c_\nu, \\ &D_\mu \Psi = (\partial_\mu + igA^a_\mu \tau_a)\Psi, \\ &G_{ab} = \frac{1}{2}\mathrm{Tr}(\tau_a \tau_b^\dagger) = \frac{1}{4N}\lambda^d_{ac}\lambda^c_{db}, \end{aligned} \tag{2..3}$$

and τ_a ($1 \leq a \leq N^2 - 1$) are the generators of $SU(N)$, λ^c_{ab} are the structure constants with respect to τ_a, $\{G_{ab}\}$ is a Riemannian metric on $SU(N)$ (see [15]), $A^a_\mu = (A^a_0, A^a_1, A^a_2, A^a_3)^T$ ($1 \leq a \leq N^2 - 1$) are the $SU(N)$ gauge fields, $\Psi = (\Psi_1, \cdots, \Psi_N)^T$ are the N Dirac spinors.

In [15], we have shown that the quantities in (2..2) and (2..3) with indices a, b, c are the $SU(N)$ representation tensors under the transformations

$$\widetilde{\tau}_a = x^b_a \tau_b, \tag{2..4}$$

where (x^b_a) is a K-th order nondegenerate matrix, and $K = N^2 - 1$.

Intuitively speaking, all $SU(N)$ gauge theories should be invariant under transformations (2..4) of different representations of $SU(N)$. This leads us to postulate PRI in [15]:

Principle of Representation Invariance (PRI). *Any $SU(N)$ gauge field theory is independent of the choice of $SU(N)$ generators $\{\tau_a\}$. Namely, the action (2..2) of gauge fields is invariant and the associated gauge field equations are covariant under the transformations (2..4).*

PRI provides a strong restriction on gauge field theories, and we address now some direct consequences of such restrictions, and we refer interested readers to [15] for more details. Here we make a few remarks.

Remark 2..1. The introduction of PRI is based purely on mathematical logic, and its validity is unquestionable. A physical theory violating PRI may only be considered as an approximation of laws of Nature.

Remark 2..2. Based on PRI, for the gauge fields A^a_μ and G^k_μ, corresponding to two different gauge groups $SU(N_1)$ and $SU(N_2)$, the following combinations

$$\alpha A^a_\mu + \beta G^k_\mu \tag{2..5}$$

are prohibited. The reason is that A^a_μ is an $SU(N_1)$-tensor with tensor index a, and G^k_μ is an $SU(N_2)$-tensor with tensor index k. The above combination violates PRI. This point of view clearly shows that the classical electroweak theory violates PRI. Of course, this remark does not undermine the importance of the classical electroweak theory, in the spirit as the comparison of the Newton mechanics and the Einstein theories of relativity.

2.2 Principle of Interaction Dynamics

Let $(M, g_{\mu\nu})$ be a Riemannian manifold of space-time with Minkowski signature, and $A = (A_1, \cdots, A_n)$ be a vector field. We define two differential operators acting on an (r, s)-tensor u as

$$\nabla_A u = \nabla u + u \otimes A, \tag{2..6}$$

$$\mathrm{div}_A u = \mathrm{div} u - A \cdot u. \tag{2..7}$$

For a functional $F(u)$, a tensor u_0 is an extremum point of F under the div_A-free constraint, if u_0 satisfies

$$\frac{d}{d\lambda} F(u_0 + \lambda X)\bigg|_{\lambda=0} = (\delta F(u_0), X) = 0 \qquad \text{for all } X \text{ with } \mathrm{div}_A X = 0.$$

In [15], we have shown that if u_0 is an extremum point of $F(u)$ with the div_A-free constraint, then there exists an $(r-1, s)$ tensor (or $(r, s-1)$ tensor) ϕ such that u_0 is a solution of the following equation

$$\delta F(u) = D_A \phi.$$

We remark that the div_A-free constraint is equivalent to energy-momentum conservation. The original motivation of div_A-free constraints was to take into consideration the

presence of dark energy and dark matter in the modified gravitational field equations [16], and then it is natural for us to postulate the following principle of interaction dynamics (PID) [14]:

Principle of Interaction Dynamics (PID). *For all physical interactions there are Lagrangian actions*

$$L(g, A, \psi) = \int_M \mathcal{L}(g_{ij}, A, \psi)\sqrt{-g}dx, \tag{2..8}$$

where $g = \{g_{ij}\}$ is the Riemann metric representing the gravitational potential, A is a set of vector fields representing gauge and mass potentials, and ψ are the wave functions of particles. The action (2..8) satisfy the invariance of general relativity (or Lorentz invariance), the gauge invariance, and PID. Moreover, the states (g, A, ψ) are the extremum points of (2..8) with the div_A*-free constraint.*

A few remarks about PID are now in order.

Remark 2..3. The original motivation for PID was to explain dark energy and dark matter, which cannot be explained in the classical Einstein gravitational field equations. The gravitational field equations based on PID

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = -\frac{8\pi G}{c^4}T_{\mu\nu} - D_\mu\Psi_\nu \tag{2..9}$$

offer a good explanation for both dark matter and dark energy; see [16] for details. Also it is worth mentioning here that the classical Einstein field equations consist of 10 equations solving 6 unknowns [13, 16], and is an overdetermined systems, which will have no solutions under certain physical conditions. The new gravitational field equations (2..9) possess the same number of unknowns and the number of equations are the same, and are mathematically more consistent.

In addition, the new term $D_\mu\Psi_\nu$ in (2..9) cannot be derived 1) from any existing $F(R)$ theories, and 2) from any existing scalar field theory.

Remark 2..4. PID offers a natural spontaneous symmetry breaking and mass generation mechanism.

2.3 Remarks on Field Theory Based on PID and PRI

With PRI and PID at our disposal, a unified field theory for interactions in Nature is developed recently in [14, 15].

Our viewpoint to derive the unified field theory is as follows. The topological and geometric structure of the space-time manifold M are determined by a few basic symmetry and dynamics principles: 1) the basic symmetry principles lead to the Riemannian metric $\{g_{\mu\nu}\}$ and the gauge fields $\{G_\mu\}$, 2) the basic dynamic principles determine the field equations for $\{g_{\mu\nu}\}$ and $\{G_\mu\}$, and 3) the field equations determine $\{g_{\mu\nu}\}$ and $\{G_\mu\}$, and consequently the structure of space-time manifold.

Also, a consistent unified field theory should obey the following requirements:

(1) Its Lagrangian action density must satisfy all known basic symmetry principles, including the principle of general relativity, the principle of Lorentz invariance, the principle of gauge invariance, and the principle of representation invariance (PRI);

(2) The Lagrangian action can be naturally decoupled to Lagrangian actions for individual interactions: the Einstein-Hilbert action for gravity, the $U(1)$ gauge field action for electromagnetism, the $SU(2)$ gauge field action for weak interaction and the $SU(3)$ gauge field action for strong interaction.

In fact, the unified field theory developed by the authors [14, 15] is uniquely determined by these two requirement; see [14, 15] for details.

3. Strong Interaction Quark Potential

3.1 Field equations for strong interaction based on PID and PRI

Both PID and PRI can be directly applied to strong interaction. In other words, the unified field model can be easily decoupled to study each individual interaction when other interactions are weak. Then the decoupled $SU(3)$ field equations, obeying both PID and PRI, are given as follows [15]:

$$\begin{aligned} G^s_{kj}\left[\partial^\nu S^j_{\nu\mu} - \frac{g_s}{\hbar c}\lambda^j_{il} g^{\alpha\beta} S^i_{\alpha\mu} S^l_\beta\right] - g_s J^s_{k\mu} & \\ = \left[\partial_\mu - \frac{1}{4}\left(\frac{m_\pi c}{\hbar}\right)^2 x_\mu + \frac{g_s\delta}{\hbar c}\alpha_k S^k_\mu\right]\phi^s_k, & \end{aligned} \tag{3..1}$$

$$i\gamma^\mu\left(\partial_\mu + i\frac{g_s}{\hbar c} S^k_\mu \tau_k\right)\psi - \frac{m_f c}{\hbar}\psi = 0, \tag{3..2}$$

where $\alpha^s_k = (\alpha^s_1, \cdots, \alpha^s_8)$ is the $SU(3)$ constant vector such that $\alpha^s_k\alpha^s_k = 1$, δ is a constant, S^k_μ $(k = 1, \cdots, 8)$ are the $SU(3)$ gauge fields, and

$$S^j_{\mu\nu} = \partial_\mu S^j_\nu - \partial_\nu S^j_\mu + \frac{g_s}{\hbar c}\lambda^j_{kl} S^k_\mu S^l_\nu, \qquad J^s_{k\mu} = \bar{\psi}\gamma_\mu\tau_k\psi, \qquad \gamma_\mu = \gamma^\mu.$$

In [15], we have shown that if the $SU(3)$ generators λ_k are taken to be the Gell-Mann matrices, then the corresponding metric $G^s_{\mu\nu}$ is Euclidian: $G^s_{\mu\nu} = \delta_{\mu\nu}$.

The above field equations introduce a natural duality:

$$S^k_\mu \longleftrightarrow \phi^k_s = \phi^s_k \qquad \text{for} \quad k = 1, \cdots, 8,$$

which represents the duality between eight gluons and their dual fields.

3.2 Strong interaction quark potential

One important point of view in the unified field theory is that each interaction has its own interaction charge, which is the source of the corresponding force. Also, each interaction possesses its own interaction potential and force formulas, given by

$$F = -g\nabla\Phi, \qquad g \text{ is the interaction charge.}$$

The physical quantities representing strong interaction potentials are the eight $SU(3)$ gauge fields S^k_μ. It is clear that the strong interaction potential must be color-independent and given by the following PRI invariant potential

$$S_\mu = \alpha_k S^k_\mu, \tag{3..3}$$

and the corresponding strong interaction potential and force are given by

$$\begin{aligned} &\Phi_s = S_0, && \text{the time component of } S_\mu,\\ &F_s = -g_s \nabla \Phi_s, && g_s \text{ is the strong charge.} \end{aligned} \tag{3..4}$$

We now derive a formula for the quark potential Φ_s from the field equations (3..1) and (3..2). We proceed in several steps as follows.

Step 1. Using the Gell-Mann matrices as the representation matrices and taking inner product of the field equations with α_k, we derive that

$$\begin{aligned} \partial^\nu S_{\nu\mu} - \frac{g_s}{\hbar c}\lambda_{ij} g^{\alpha\beta} S^i_{\alpha\mu} S^j_\beta - g_s J^s_\mu & \\ = \left[\partial_\mu - \frac{1}{4}\left(\frac{mc}{\hbar}\right)^2 x_\mu + \frac{g_s \delta}{\hbar c} S_\mu\right]\phi_s&, \end{aligned} \tag{3..5}$$

$$\begin{aligned} \left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \triangle\right)\phi_s + \left(\frac{mc}{\hbar}\right)^2 \phi_s + \frac{1}{4}\left(\frac{mc}{\hbar}\right)^2 x_\mu \partial^\mu \phi_s & \\ = g_s \partial^\mu J^s_\mu + \frac{g_s}{\hbar c}\partial^\mu(\lambda_{ij} g^{\alpha\beta} S^i_{\alpha\mu} S^j_\beta - \delta S_\mu \phi_s)&, \end{aligned} \tag{3..6}$$

where

$$\begin{aligned} &\lambda_{ij} = \alpha_k \lambda^k_{ij}, && m = m_\pi,\\ &\gamma_\mu = \gamma^\mu, && \tau_k = \tau^k,\\ &J^s_\mu = \alpha_k \bar{\psi}\gamma_\mu \tau_k \psi, && S_{\mu\nu} = \partial_\mu S_\nu - \partial_\nu S_\mu + \frac{g_s}{\hbar c}\lambda_{ij} S^i_\mu S^j_\nu. \end{aligned} \tag{3..7}$$

Based on the superposition property of the quark potential, $\Phi_s = S_0$ and ϕ_s obey a linear relationship. Namely, we can choose the supplement equations properly, which are due to the introduction of the dual fields, such that the $\mu = 0$ equations of (3..5) and (3..6) are linear. In other words, the supplement equations contain the following two equations:

$$\begin{aligned} &\lambda_{ij}\left[\partial^\nu(S^i_\nu S^j_0) - g^{\alpha\beta} S^i_{\alpha 0} S^j_\beta\right] + \delta S_0 \phi_s = 0,\\ &\partial^\mu\left[\lambda_{ij} g^{\alpha\beta} S^i_{\alpha\mu} S^j_\beta - \delta S_\mu \phi_s\right] = 0. \end{aligned} \tag{3..8}$$

Also, physically, we take two more supplement equations

$$\begin{aligned} &x_\mu \partial^\mu \phi_s = 0,\\ &\partial^\mu S_\mu = 0, \end{aligned} \tag{3..9}$$

together with the following static assumption:

$$\frac{\partial S_0}{\partial t} = 0, \qquad \frac{\partial \phi_s}{\partial t} = 0.$$

With the above gauge fixing conditions and static assumption, we derive equations for the quark potential ($\mu = 0$) and the dual potential as follows:

$$-\triangle\Phi_s = g_s Q - \frac{1}{4}k_0^2 c\tau\phi_s, \tag{3..10}$$

$$-\triangle\phi_s + k_0^2\phi_s = g_s\partial^\mu J_\mu^s, \tag{3..11}$$

where $x_0 = -c\tau$ is the wave length of ϕ_s, $k_0 = mc/\hbar$, $Q = -J_0^s$. For quarks, we have

$$k_0 = \frac{mc}{\hbar} \sim 10^{16}\text{cm}^{-1}, \qquad k_0 c\tau \geq 10^5. \tag{3..12}$$

Step 2. Solution of (3..11). By definition, we have

$$\partial^\mu J_\mu^s = \alpha_k\partial_\mu\bar{\psi}\gamma^\mu\tau_k\psi + \alpha_k\bar{\psi}\gamma^\mu\tau_k\partial_\mu\psi.$$

In view of the Dirac equation (3..2),

$$\partial_\mu\bar{\psi}\gamma^\mu\tau_k\psi = i\frac{g_s}{\hbar c}S_\mu^j\bar{\psi}\gamma^\mu\tau_j\tau_k\psi + i\frac{m_f c}{\hbar}\bar{\psi}\tau_k\psi,$$
$$\bar{\psi}\gamma^\mu\tau_k\partial_\mu\psi = -i\frac{g_s}{\hbar c}S_\mu^j\bar{q}\gamma^\mu\tau_k\tau_j\psi - i\frac{m_f c}{\hbar}\bar{\psi}\tau_k\psi.$$

Hence we arrive at

$$\partial^\mu J_\mu^s = \frac{ig_s}{\hbar c}\alpha_k S_\mu^j\bar{\psi}\gamma^\mu[\tau_j,\tau_k]\psi = -\frac{2g_s}{\hbar c}\alpha_k S_\mu^j\lambda_{jk}^i J_i^{s\mu}.$$

Since $J_i^{s\mu} = J_\mu^{si}$ is the quark current density, we have

$$\alpha_k\lambda_{jk}^i J_i^{s\mu} = \theta_j^\mu\delta(r),$$

where $\delta(r)$ is the Dirac delta function, θ_j^μ is a constant tensor, inversely proportional to the volume of a quark. Hence

$$\alpha_k S_\mu^j\lambda_{jk}^i J_i^{s\mu} = \bar{S}_\mu^j\theta_j^\mu\delta(r), \tag{3..13}$$

where $\bar{S}_\mu^j \sim \bar{S}_\mu^j(0)$ takes the following average value

$$\bar{S}_\mu^j = \frac{1}{|B_{\rho_0}|}\int_{B_{\rho_0}} S_\mu^j dV.$$

Here ρ_0 is the radius of a quark. Later, we shall see that

$$S_\mu^j \sim \frac{1}{r} \text{ as } r \to \infty.$$

Hence we deduce that

$$\bar{S}_\mu^j = \xi_\mu^j\rho_0^{-1} \qquad (\xi_\mu^j \text{ is a constant tensor}). \tag{3..14}$$

Finally, we arrive at

$$\partial^\mu J^s_\mu = -\frac{\kappa\delta(r)}{\rho_0} \qquad (\rho_0 \text{ is the radius of a quark}). \tag{3..15}$$

where κ is a constant given by

$$\kappa = \frac{2g_s}{\hbar c}\xi^j_\mu \theta^\mu_j \quad \text{inversely proportional to the volume } V_q \text{ of a quark.} \tag{3..16}$$

Therefore, equation (3..11) is rewritten as

$$-\triangle\phi_s + k_0^2\phi_s = -\frac{g_s\kappa\delta(r)}{\rho_0}, \tag{3..17}$$

whose solution is given by

$$\phi_s = -\frac{g_s\kappa}{\rho_0}\frac{1}{r}e^{-k_0 r}. \tag{3..18}$$

STEP 3. SOLUTION OF (3..10). The quantity $g_s Q = -g_s J^s_0$ is the strong charge density of a quark, and without loss of generality, we assume that

$$Q = \beta\delta(r), \tag{3..19}$$

and $\beta > 0$ is a constant, inversely proportional to the quark volume. Hence (3..10) can be rewritten as

$$-\triangle\Phi_s = g_s\beta\delta(r) + \frac{g_s A}{\rho_0}\frac{1}{r}e^{-k_0 r}, \tag{3..20}$$

where A is a constant given by

$$A = \frac{k_0^2 c\tau\kappa}{4} \quad \text{with physical dimension } \frac{1}{L}. \tag{3..21}$$

Assume $\Phi_s = \Phi_s(r)$ is radially symmetric, then (3..20) can be rewritten as

$$-\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d}{dr}\right)\Phi_s = g_s\beta\delta(r) + \frac{g_s A}{\rho_0}\frac{1}{r}e^{-k_0 r},$$

whose solution takes the form

$$\Phi_s = g_s\left[\frac{\beta}{r} - \frac{A}{\rho_0}\varphi(r)e^{-k_0 r}\right], \tag{3..22}$$

where φ solves

$$\varphi'' + 2\left(\frac{1}{r} - k_0\right)\varphi' - \left(\frac{2k_0}{r} - k_0^2\right)\varphi = \frac{1}{r}. \tag{3..23}$$

Assume that the solution φ of (3..23) is given by

$$\varphi = \sum_{k=0}^{\infty}\alpha_k r^k. \tag{3..24}$$

Inserting φ in (3..23) and comparing the coefficients of r^k, we obtain the relations

$$\begin{aligned} \alpha_1 &= k_0\alpha_0 + \frac{1}{2}, \\ \alpha_2 &= \frac{1}{2}k_0^2\alpha_0 + \frac{1}{3}k_0, \\ &\vdots \\ \alpha_k &= \frac{2k_0}{k+1}\alpha_{k-1} - k_0^2\alpha_{k-2} \quad \text{for } k \geq 2, \end{aligned} \tag{3..25}$$

where α_0 is a free parameter with dimension L. Hence

$$\varphi(r) = \alpha_0 \left(1 + k_0 r + \frac{r}{2\alpha_0} + o(r)\right),$$

and often it is sufficient to take a first-order approximation.

Step 4. Strong interaction quark potential. Formula (3..22) provides an accurate strong interaction quark potential formula:

$$\Phi_q = g_s\beta_q \left[\frac{1}{r} - \frac{A_q}{\rho_0}\varphi(r)e^{-k_0 r}\right], \tag{3..26}$$

where $A_q = \alpha_0 A$ is a dimensionless parameter, $k_0 = 1/r_0$, r_0 is the average radius of hadrons, and $\varphi(r)$ is a dimensionless function, which can approximately be given by

$$\varphi(r) = 1 + \frac{r}{r_0}. \tag{3..27}$$

Namely, there are three constants in (3..26), A_q, β_q, and r_0. Also,

$$\beta_q \text{ is inversely proportional to the quark volume } V_q. \tag{3..28}$$

For simplicity, we can take $\beta_q = 1, \varphi = 1$, then (3..26) takes the following form

$$\Phi_q = g_s \left[\frac{1}{r} - \frac{A_q}{\rho_0}e^{-k_0 r}\right].$$

4. Layered Formulas of Strong Interaction Potentials

Different from gravity and electromagnetic force, strong interaction is short-ranged with different strengths in different levels. For example, in the quark level, strong interaction confines quarks inside hadrons, in the nucleon level, strong interaction bounds nucleons inside atoms, and in the atom and molecule level, strong interaction almost diminishes. This layered phenomena can be well-explained using the unified field theory based on PID and PRI. We derive in this section strong interaction potentials in different levels.

We start with strong interaction nucleon potential. For strong interaction of nucleons, we still use the $SU(3)$ QCD action

$$\mathcal{L} = -\frac{1}{4}S^k_{\mu\nu}S^{k\mu\nu} + \hbar c\bar{n}\left(i\gamma^\mu D_\mu - \frac{mc}{\hbar}\right)n, \tag{4..1}$$

where S^k_μ are the strong interaction gauge fields,

$$n = (a_1 n_0, a_2 n_0, a_3 n_0), \qquad a_1^2 + a_2^2 + a_3^2 = 1, \tag{4..2}$$

n_0 is the wave function of a nucleon, and

$$D_\mu n = (\hbar c \partial_\mu + \frac{i g_s}{\hbar c} S^k_\mu \tau_k) n. \tag{4..3}$$

The action $\mathcal{L}$ defined by (4..1) is $SU(3)$ gauge invariant. Physically, this means that nucleons are indistinguishable in strong interaction. With PID, the corresponding field equations corresponding to the action $\mathcal{L}$ are

$$\partial^\nu S^k_{\nu\mu} + \frac{g_s}{\hbar c} \lambda^k_{ij} g^{\alpha\beta} S^i_{\alpha\mu} S^j_\beta - g_s Q^k_\mu = \left(\partial_\mu - \frac{k_1^2}{4} x_\mu\right) \phi^k_n, \tag{4..4}$$

$$i\gamma^\mu \left(\partial_\mu + i\frac{g_s}{\hbar c} S^k_\mu \tau_k\right) n - \frac{mc}{\hbar} n = 0, \tag{4..5}$$

where the parameter k_1 is defined by

$$r_1 = \frac{1}{k_1} = 10^{-13} \text{ cm is the Yukawa radius}, \tag{4..6}$$

which is the average radius of nucleons, and

$$Q^k_\mu = \bar{n} \gamma_\mu \tau^k n \qquad (\lambda^k = \lambda_k). \tag{4..7}$$

Let the nucleon strong interaction Φ_n and its dual potential ϕ_n be defined by

$$\Phi_n = \alpha_k S^k_0, \qquad \phi_n = \alpha_k \phi^k_n.$$

In the same spirit as for deriving the quark potential (3..26), we deduce that

$$\begin{aligned} & -\triangle \Phi_n = g_s Q_0 - \frac{1}{4} k_1^2 c \tau_1 \phi_n, \\ & -\triangle \phi_n + k_1^2 \phi_n = g_s \partial^\mu Q_\mu, \end{aligned} \tag{4..8}$$

where $c\tau_1$ is the wave length of ϕ_n, and

$$Q = \alpha_k Q^k_\mu = (Q_0, Q_1, Q_2, Q_3)$$

represents nucleon current density. Similar to (3..15) and (3..19), we have

$$\partial^\mu Q_\mu = -\frac{k_n \delta(r)}{\rho_n}, \qquad Q_0 = \beta_n \delta(r), \tag{4..9}$$

where ρ_n is the radius of a nucleon, β_n and k_n are constants, inversely proportional to the volume of nucleons. Consequently, we derive the following strong nucleon potential:

$$\Phi_n = \beta_n g_s \left[\frac{1}{r} - \frac{A_n}{\rho_n} \varphi(r) e^{-k_1 r}\right], \tag{4..10}$$

where $\varphi(r)$ is as (3..24) and (3..25), and

$$A_n = \frac{\kappa_n k_1^2 c\tau_1}{4\beta_n}.$$

Note that β_q and β_n are inversely proportional to the volumes V_q and V_n, respectively. Hence

$$\frac{\beta_n}{\beta_q} = \frac{NV_q}{V_n} = N\left(\frac{\rho_0}{\rho_n}\right)^3, \qquad N = 3.$$

Here $N = 3$ is the number of strong charges in a nucleon. We then conclude that the strong interaction nucleon potential can be rewritten as (always assuming $\beta_q = 1$):

$$\Phi_n = 3g_s\left(\frac{\rho_0}{\rho_n}\right)^3\left[\frac{1}{r} - \frac{A_n}{\rho_n}\varphi(r)e^{-k_1 r}\right].$$

In summary, for a particle with N strong charges and radius ρ, the strong interaction potential can be written as

$$\Phi = Ng_s\left(\frac{\rho_0}{\rho}\right)^3\left[\frac{1}{r} - \frac{A}{\rho}\varphi(r)e^{-kr}\right], \tag{4..11}$$

where ρ_0 is the quark radius, A is a dimensionless constant depending on the particle, $1/k$ is the radius of strong attraction, and $\varphi(r) = 1 + kr$.

More specifically, we, respectively, obtain the strong quark potential Φ_q, the strong nucleon potential Φ_n and the strong atom/molecule potential Φ_a as follows:

$$\begin{aligned}
\Phi_q &= g_s\left[\frac{1}{r} - \frac{A_q}{\rho}(1 + k_0 r)e^{-k_0 r}\right], \\
\Phi_n &= 3g_s\left(\frac{\rho_0}{\rho_n}\right)^3\left[\frac{1}{r} - \frac{A_n}{\rho_n}(1 + k_1 r)e^{-k_1 r}\right], \\
\Phi_a &= 3Ng_s\left(\frac{\rho_0}{\rho_a}\right)^3\left[\frac{1}{r} - \frac{A_a}{\rho_a}(1 + k_1 r)e^{-k_1 r}\right],
\end{aligned} \tag{4..12}$$

where N is the number of nucleons in an atom/molecule. These formulas provide strong interaction potentials for particles in different levels.

Remark 4..1. 1. We remark here that the strong charge g_s used in this article is to use quark as a reference level. If we use weaktons as the reference level, the layered strong interaction potentials are given by

$$\Phi = g_s(\rho)\left[\frac{1}{r} - \frac{A_s}{\rho}\left(1 + \frac{r}{R}\right)e^{-r/R}\right], \tag{4..13}$$

where A_s is a constant depending on the particle type, and R is the attracting radius of strong interactions given by

$$R = \begin{cases} 10^{-16}\text{cm} & \text{for } w^* \text{ and quarks,} \\ 10^{-13}\text{cm} & \text{for hadrons.} \end{cases}$$

Also, the strong charge is defined by

$$g_s(\rho) = \left(\frac{\rho_w}{\rho}\right)^3 g_s, \tag{4..14}$$

where g_s is the strong charge using weakton as a reference level. In other words, the g_s in (4..14) is equivalent to

$$\left(\frac{\rho_0}{\rho_w}\right)^3 g_s,$$

where g_s is the strong charge used in (4..12) and throughout this article.

2. The strong interaction potential between two particles with N_1, N_2 charges g_s and radii ρ_1, ρ_2 is given by

$$\Phi_s = N_1 N_2 g_s(\rho_1) g_s(\rho_2) \left[\frac{1}{r} - \frac{\bar{A}_s}{\sqrt{\rho_1\rho_2}}\left(1 + \frac{r}{R}\right) e^{-r/R}\right], \tag{4..15}$$

where $g_s(\rho_k)$ $(k = 1, 2)$ are as in (4..14), and $\bar{A}_s$ is a constant depending on the types of two particle involved.

5. Quark Confinement and Asymptotic Freedom

With the strong interaction potential formulas at our disposal, we are ready to derive quark confinement and asymptotic freedom for strong interactions, two important phenomena in particle physics.

5.1 Quark confinement

We start with quark confinement. The strong interaction bounding energy E of two particles is given by

$$E = g_s \Phi(r), \qquad \Phi(r) \text{ is given by (4..11).}$$

Based on the quark potential (3..26), there are two radii: the quark radius ρ_0 and the quark attraction radius $r_0 = 1/k_0$:

$$\rho_0 \sim 10^{-19} \text{ cm}, \qquad r_0 \sim 10^{-16} \text{ cm}. \tag{5..1}$$

Hence we derive from (3..28) the strong force between quarks:

$$F_q = -g_s \frac{d\Phi_q}{dr} = g_s^2 \left[\frac{1}{r^2} - \frac{A_1}{\rho_0}\frac{1}{r_0} e^{-k_0 r}\right], \tag{5..2}$$

which implies that there are two radii $\bar{r}_1$ and $\bar{r}_2$ such that $\rho_0 < \bar{r}_1 < r_0 < \bar{r}_2$ and

$$F_q \begin{cases} > 0 & \text{for } 0 < r < \bar{r}_1, \\ < 0 & \text{for } \bar{r}_1 < r < \bar{r}_2. \end{cases} \tag{5..3}$$

It is clear then that in the region where $\bar{r}_1 < r < \bar{r}_2$, strong force between quarks is attractive. In particular, the largest attraction force between quarks is of the following order:

$$F \sim \frac{1}{\rho_0},$$

which indicates that the largest strong attractive force between quarks $F \to \infty$ if $\rho_0 \to 0$. This is the reason for quark confinement.

The quark confinement can also be better explained from the viewpoint of the strong quark bounding energy E_q and the hadronic bounding energy E_n. We have

$$\frac{E_q}{E_n} = \frac{A_q}{\rho_0} e^{(k_1 - k_0)r} \Big/ \left(3 \left(\frac{\rho_0}{\rho_n} \right)^6 \frac{A_n}{\rho_n} \right) \sim \left(\frac{\rho_n}{\rho_0} \right)^6 \tag{5..4}$$

Based on physical estimates of $\rho_n = 10^{-15}$cm and $\rho_0 = 10^{-19}$cm, we derive that

$$E_q \sim 10^{24} E_n. \tag{5..5}$$

We know that $E_n \sim 10^{-2}$GeV, then we obtain that

$$E_q \geq 10^{22}\text{GeV}.$$

This clearly shows that there is no observed free quarks and the quark is confined in hadrons.

5.2 Asymptotic freedom

The strong interaction potentials provide also a natural explanation of the asymptotic freedom phenomena. By (5..3), we see that

$$F_q \sim 0 \qquad \text{near } r = \bar{r}_1.$$

This indicates that there is a free shell region inside a proton, for example, with radius $\widetilde{r}$, such the three quarks are free in this shell region.

When a low energy electron collides with the proton, the electromagnetic force causes the electron moving away, leading to the elastic scattering

$$e^- + p \to e^- + p.$$

However, when a high speed electron collides with a proton, it can run into the inside of the proton, interacting with one of the quarks. Since the quark was in a free shell region with no force acting upon it, this particular quark will behavior as a free quark. As it moves into the attracting region of the proton, the quark confinement will hold this quark, which, at the same time, will collide with gluons, exchanging quarks, leading to the following inelastic scatterings:

$$e^- + p \to e^- + p + \pi^0,$$
$$e^- + p \to e^- + n + \pi^+.$$

This explains the asymptotic freedom.

6. Yukawa potential and Short-Range Nature of Strong Interaction

One of the mysteries of the strong interaction is the different characteristics exhibited in the quark level and in the nucleon level. In the quark level,

$$\text{quark strong interaction at } r = 10^{-16}\text{cm is infinitely } \textit{attractive}; \tag{6..1}$$

while in the nucleon level,

$$\text{nucleon strong interaction for } 0 < r < 10^{-13}\text{cm is } \textit{repulsive}. \tag{6..2}$$

The different characteristics of the strong interactions demonstrated in (]ref6.1r) and (6..2) can hardly explained by any existing theory. However, the layered strong interaction potentials in (4..12) derived based on PID and PRI lead to a natural explanation of these characteristics, as well as explanations of the quark confinement and asymptotic freedom in the previous section. In this section, we shall explain the Yukawa force between nucleons and the short-range nature of strong interaction.

6.1 Modified Yukawa potential based on PID and PRI

Based on the classical strong interaction theory, the potential bounding hadrons for nucleons is the Yukawa potential:

$$\Phi_Y = -\frac{g}{r}e^{-k_1 r}, \qquad r_1 = \frac{1}{k_1} = 10^{-13}\text{cm}, \tag{6..3}$$

where g is the meson charge. The corresponding Yukawa strong force is

$$F_Y = -g\frac{d\Phi_Y}{dr} = -g^2\left(\frac{1}{r^2} + \frac{1}{r_1 r}\right)e^{-k_1 r} < 0 \qquad \text{for } r > 0, \tag{6..4}$$

indicating that the strong nucleon force is always attractive. However, experimentally, we know [27] that

$$F_{\text{experiment}} \begin{cases} > 0 & \text{for } 0 < r < \bar{r} = \frac{1}{2} \times 10^{-13}\text{cm}, \\ < 0 & \text{for } \bar{r} < r. \end{cases} \tag{6..5}$$

The discrepancy between (6..4) and (6..5) shows that the Yukawa potential is valid only in a small attractive range of strong nucleon force.

The derived nucleon potential Φ_n in (4..12) is schematically shown in Figure 6..1, which is consistent with the experimental results; see [27]. Also the corresponding strong nucleon force is given by

$$F_n = -g_s\frac{d\Phi_n}{dr} = 3g_s^2\left(\frac{\rho_0}{\rho_n}\right)^6\left[\frac{1}{r^2} - \frac{A_n}{\rho_n}\frac{r}{r_1^2}e^{-r/r_1}\right], \tag{6..6}$$

where

$$\rho_n = 10^{-15}\text{cm}, \qquad r_1 = 10^{-13}\text{cm}. \tag{6..7}$$

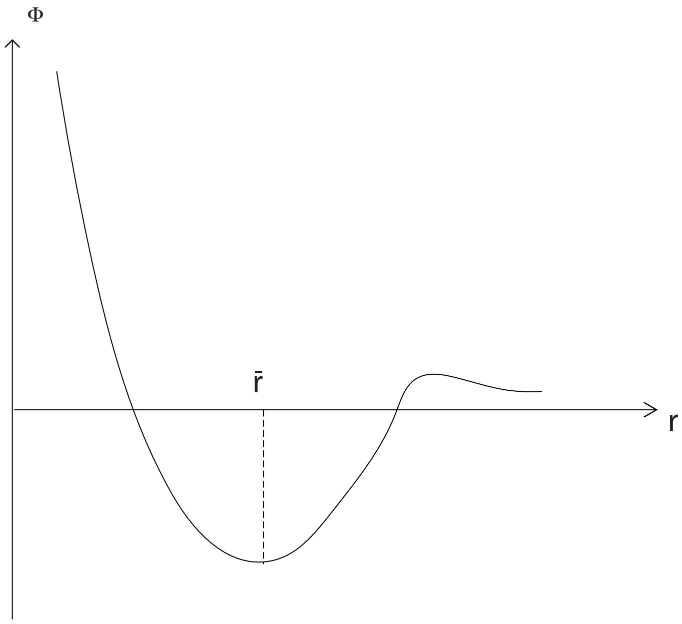

Fig. 6..1

By (6..6) and (6..7), we can determine A_n from the experimental value $\bar{r}$ in (6..5) as follows.

Set $F_n = 0$, then we have

$$r^3 e^{-r/r_1} = \frac{\rho_n r_1^2}{A_n}. \tag{6..8}$$

Using $\bar{r} = r_1/2$, this formula shows that

$$A_n = 8e^{1/2} \times 10^{-3}. \tag{6..9}$$

In addition, by (6..8) and (6..9), we derive the two zeros $\bar{r}$ and r_2 of F_n:

$$\bar{r} = \frac{r_1}{2}, \qquad r_2 \sim 9r_1. \tag{6..10}$$

In addition, using the magnitude of the Yukawa force and the magnitude of the attractive force of F_n are given by

$$F_Y \sim -2g^2/r_1^2, \qquad F_n \sim -3g_s^2 \left(\frac{\rho_0}{\rho_n}\right)^6 \frac{A_n}{\rho_n} \frac{r_1}{r_1^2},$$

which lead to

$$3g_s^2 \left(\frac{\rho_0}{\rho_n}\right)^6 \frac{A_n r_1}{\rho_n} = 2g^2 \qquad (g^2 \sim 10\hbar c). \tag{6..11}$$

6.2 Physical conclusions on nucleon strong force

The discussion in the previous subsection leads to the following physical conclusions:

(1) The modified Yukawa formula based on PID and PRI is

$$F_n = g^2 \left[\frac{1}{4\sqrt{e}} \frac{1}{r^2} - \frac{2r}{r_1^3} e^{-r/r_1} \right], \tag{6..12}$$

where $g^2 = 10\hbar c$ is the usual nucleon interaction constant, and e is the base of nature logarithm.

(2) The attraction and repelling regions of the strong nucleon force F_n are

$$F_n \begin{cases} > 0 & \text{for } 0 < r < r_1/2, \\ < 0 & \text{for } r_1/2 < r < 9r_1, \\ > 0 & \text{for } 9r_1 < r, \end{cases}$$

where $r_1 = 10^{-13}$cm.

(3) The largest attraction of F_n is achieved at $r \sim 1.5r_1$ with value given by

$$F_{\max} = -\frac{g^2}{\sqrt{e} r_1^2}.$$

(4) By (6..11), the value of the strong charge g_s is given by

$$g_s^2 = \frac{1}{2} \times 10^{25} \hbar c.$$

(5) The strong nucleon interaction constant A_n is given by

$$A_n = 8\sqrt{e} \times 10^{-3}.$$

6.3 Short-range nature of strong interaction

By the layered potentials for strong interactions, we see that when the nucleons form an atom, the nucleon potential is no longer valid, and the correct potential becomes the strong interaction potential for atoms given by the third formula in (4..12). The corresponding force formula is given by

$$F_a = 9N^2 g_s^2 \left(\frac{\rho}{\rho_a} \right)^6 \left[\frac{1}{r^2} - \frac{A_a}{\rho_a} \frac{r}{r_1^2} e^{-r/r_1} \right], \tag{6..13}$$

where

$$\text{for atom: } \rho_a = 10^{-8}\text{cm}, \qquad \text{for molecule: } \rho_a = 10^{-7}\text{cm}. \tag{6..14}$$

Also, bounding force between atoms and molecules are electromagnetic force with strength given by

$$\frac{e^2}{\hbar c} = \frac{1}{137}. \tag{6..15}$$

where e is the electric charge. Hence at the atom/molecule scales, the ratio between strong force and electromagnetic attraction force is

$$\frac{F_a}{F_e} = 9N^2 g_s^2 \left(\frac{\rho_0}{\rho_a}\right)^6 \Big/ N^2 e^2 = 9g_s^2 \left(\frac{\rho_0}{\rho_a}\right)^6 \Big/ e^2, \tag{6..16}$$

where in the last term, the first e is the base of the natural logarithm, and the second e is the electric charge. Consequently,

$$\frac{F_a}{F_e} \sim \begin{cases} 10^{-38} & \text{at the atomic level,} \\ 10^{-44} & \text{at the molecular level.} \end{cases} \tag{6..17}$$

This clearly demonstrates the short-range nature of strong interaction.

7. Conclusions

This paper addresses a number of important consequences of the duality induced from the new field equations for strong interaction derived in [15] based on two basic principles PID and PRI. First, one prediction of this duality is the existence of eight dual gluon scalar fields. From the field theoretical point of view, these fields are needed to produce the strong attraction force, leading to quark confinement.

Second, with the duality, we derive three levels of strong interaction potentials: the quark potential S_q, the nucleon/hadron potential S_n and the atom/molecule potential S_a. These potentials clearly demonstrate many features of strong interaction consistent with experimental observations. In particular, these potentials offer a clear mechanism for both quark confinement and asymptotic freedom.

Third, in the nuclear level, the new potential is an improvement of the Yukawa potential. As the distance between two nucleons is increasing, the nuclear force corresponding to the nucleon potential S_n behaves as repelling, then attracting, then repelling again and diminishes. This is exactly the picture that the experimental observations tell us. It is worth mentioning that the classical Yukawa potential can not be derived from classical QCD, while the new potentials are indeed derived directly from the new field equations.

Finally, this paper is part of a research program on unified field theory for interactions in nature. Among other features the unified field model derived in [14, 15] can be easily decoupled to study individual interactions.

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Nonzero θ_{13} and Neutrino Masses from Modified Tribimaximal Mixing TBM

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Abstract: In order to accommodate nonzero and relatively large of mixing angle θ_{13}, we modified the tribimaximal mixing(TBM) matrix by introducing a simple perturbation matrix to perturb TBM matrix. The modified TBM can reproduce nonzero mixing angle $\theta_{13} = 7.9^0$ which is in agreement with the present experimental results. By imposing two zeros texture into the obtained neutrino mass matrix from modified TBM, we then have the neutrino mass spectrum in normal hierarchy. Some phenomenological implications are also discussed.

Keywords: Neutrino Masses; Perturbation Matrix; Modified Tribimaximal Mixing(TBM)
PACS (2010): 14.60.Lm; 14.60.St; 14.60.Pq; 12.15.-y

1. Introduction

There are three types of the well-known neutrino mixing matrices; tribimaximal (TBM), bimaximal (BM), and democratic DC). These three neutrino mixing matrices patterns predict the mixing angle $\theta_{13} = 0$. Recently, the evidence of nonzero θ_{13} due to the achievement of experimental methods and tools, the assumption that the value of mixing angle θ_{13} is very small and tend to zero must be corrected or even ruled out. Concerning with the well-known mixing matrix, especially tribimaximal neutrino mixing matrix, Ishimori and Ma [1] stated explicitly that the tribimaximal mixing matrix may be dead due to the experimental fact that mixing angle θ_{13} is not zero. The nonzero and relatively large mixing angle θ_{13} have already been reported by MINOS [2], Double Chooz [3], T2K [4], Daya Bay [5], and RENO [6] collaborations.

The evidence of nonzero and relatively large θ_{13} as reported by many collaborations, several authors have already proposed some methods and models in order to explain the

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existence of nonzero θ_{13}. The simple way to accommodate a nonzero θ_{13} is to modify the neutrino mixing matrix by introducing a perturbation matrix into known mixing matrix such that it can produces a nonzero θ_{13} [7, 8, 9], breaking the scaling ansatz [10], and the other is to build the model by using some discrete symmetries [11, 12, 13, 14].

In this paper we modify TBM mixing matrix by introducung a simple perturbation matrix and calculate the mixing angle θ_{13} by using the advantages of the mixing angles θ_{21} and θ_{32} from the experimental results. The modofied TBM is used to construct the neutrino mass matrix and we evaluate the neutrino mass and its hierarchy. This paper is organized as follow: in section 2, we modify tribimaximal mixing matrix by introducing a simple perturbation matrix. In section 3, we determine the neutrino mass spectrum from modified tribimaximal mixing matrix. Finally, section 4 is devoted to conclusion.

2. Nonzero θ_{13} from the modified tribimaximal mixing matrix

The TBM mixing matrix existence is due to the experimental facts that mixing of flavors do exist in the leptonic sector especially in neutrino sector as well as in the quarks sector. The neutrino eigenstates in flavor basis $(\nu_e, \nu_\mu, \nu_\tau)$ relate to the eigenstates of neutrino in mass basis (ν_1, ν_2, ν_3) as follow:

$$\nu_i = V_{ij}\nu_j, \tag{1}$$

where $V_{ij}(i = e, \mu, \tau; j = 1, 2, 3)$ are the elements of neutrino mixing matrix. The mixing matrix V can be parameterized as follow:

$$V = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\phi} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\phi} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\phi} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\phi} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\phi} & c_{23}c_{13} \end{pmatrix} \tag{2}$$

where c_{ij} is the $\cos\theta_{ij}$, s_{ij} is the $\sin\theta_{ij}$, and θ_{ij} are the mixing angles.

One of the well-known neutrino mixing matrix (V) is the tribimaximal neutrino mixing matrix (V_{TBM}) which given by [15, 16, 17, 18, 19, 20]:

$$V_{TBM} = \begin{pmatrix} \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \end{pmatrix}. \tag{3}$$

As one can see from Eq. (3) that the entry $V_{e3} = 0$ which imply that the mixing angle θ_{13} must be zero in the tribimaximal mixing matrix. However, the latest result from long baseline neutrino oscillation experiment T2K indicates that θ_{13} is relatively large. For a vanishing Dirac CP-violating phase, the T2K collaboration reported that the values of θ_{13} are [4]:

$$5.0^o \leq \theta_{13} \leq 16.0^o, \text{and } 5.8^o \leq \theta_{13} \leq 17.8^o, \tag{4}$$

for neutrino mass in norma (NH)l and inverted (IH) hierarchies respectively, and the current combined world data[21]-[22]:

$$\Delta m_{21}^2 = 7.59 \pm 0.20(^{+0.61}_{-0.69}) \times 10^{-5} \text{ eV}^2, \tag{5}$$

$$\Delta m_{32}^2 = 2.46 \pm 0.12(\pm 0.37) \times 10^{-3} \text{ eV}^2, \text{ (for NH)} \tag{6}$$

$$\Delta m_{32}^2 = -2.36 \pm 0.11(\pm 0.37) \times 10^{-3} \text{ eV}^2, \text{ (for IH)} \tag{7}$$

$$\theta_{12} = 34.5 \pm 1.0(^{3.2}_{-2.8})^o, \quad \theta_{23} = 42.8^{+4.5}_{-2.9}(^{+10.7}_{-7.3})^o, \quad \theta_{13} = 5.1^{+3.0}_{-3.3}(\leq 12.0)^o, \tag{8}$$

at 1σ (3σ) level. The latest experimental result on θ_{13} is reported by Daya Bay Collaboration which gives [5]:

$$\sin^2 2\theta_{13} = 0.092 \pm 0.016(\text{stat.}) \pm 0.005(\text{syst.}), \tag{9}$$

and RENO Collaboration reported that [6]:

$$\sin^2 2\theta_{13} = 0.113 \pm 0.013(\text{stat.}) \pm 0.014(\text{syst.}). \tag{10}$$

Modification of neutrino mixing matrix, by introducing a perturbation matrices into neutrino mixing matrices in Eq. (3), is the easiest way to obtain the nonzero θ_{13}. The value of θ_{13} can be obtained in some parameters that can be fitted from experimental results. In this paper, the modified neutrino mixing matrices to be considered are given by:

$$V_{\text{TBM}}^{'} = V_{\text{TBM}} V_y, \tag{11}$$

where V_y is the perturbation matrices to the neutrino mixing matrices. We take the form of the perturbation matrices as follow:

$$V_y = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_y & s_y \\ 0 & -s_y & c_y \end{pmatrix}. \tag{12}$$

where c_y is the $\cos y$, and s_y is the $\sin y$.

By inserting Eqs. (3) and (12) into Eqs. (11), we then have the modified neutrino mixing matrices as follow:

$$V_{\text{TB}}^{'} = \begin{pmatrix} \frac{\sqrt{6}}{3} & \frac{\sqrt{3}}{3} c_y & \frac{\sqrt{3}}{3} s_y \\ -\frac{\sqrt{6}}{6} & \frac{\sqrt{3}}{3} c_y - \frac{\sqrt{2}}{2} s_y & \frac{\sqrt{3}}{3} s_y + \frac{\sqrt{2}}{2} c_y \\ -\frac{\sqrt{6}}{6} & \frac{\sqrt{3}}{3} c_y + \frac{\sqrt{2}}{2} s_y & \frac{\sqrt{3}}{3} s_y - \frac{\sqrt{2}}{2} c_y \end{pmatrix}, \tag{13}$$

By comparing Eqs. (13) with the neutrino mixing in standard parameterization form as shown in Eq. (2) with $\varphi = 0$, then we obtain:

$$\tan\theta_{12} = \left|\frac{\sqrt{2} c_y}{2}\right|, \quad \tan\theta_{23} = \left|\frac{\frac{\sqrt{3}}{3} s_y + \frac{\sqrt{2}}{2} c_y}{\frac{\sqrt{3}}{3} s_y - \frac{\sqrt{2}}{2} c_y}\right|, \quad \sin\theta_{13} = \left|\frac{\sqrt{3}}{3} s_y\right|. \tag{14}$$

From Eq. (14) it is apparent that for $y \to 0$, the value of $\tan\theta_{12} \to \sqrt{2}/2$ and $\tan\theta_{23} \to 1$ which imply that $\theta_{12} \to 35.264^o$ and $\theta_{23} \to 45^o$. From Eq. (14), one can see that it is possible to determine the value y and therefore the value of θ_{13} by using the experimental values of θ_{12} and θ_{23} in Eq. (8).

By inserting the experimental values of θ_{12} and θ_{23} in Eq. (8) into Eq. (14), we obtain the relations:

$$c_y = -0.03167630078 s_y, \tag{15}$$

when we use θ_{23}, and

$$c_y = 0.9713265692, \tag{16}$$

when we use θ_{12}. From both Eqs. (15) and (16), we can see that the realistic value for c_y is the value c_y in Eq. (16) that is $y = 13.7537^o$. It means that in this modification scenario, only the experimental mixing angle θ_{12} related to the mixing angle θ_{13}. From Eq. (16), we have:

$$\sin\theta_{13} = 0.137265, \tag{17}$$

that imply the mixing angle $\theta_{13} = 7.89^o$ which is in agreement with the T2K [4] and Daya Bay experimental results [5].

3. Neutrino masses from modified tribimaximal mixing matrix

We construct the neutrino mass matrix M_ν in flavor eigenstates basis (where the charged lepton mass matrix is diagonal). In this basis, the neutrino mass matrix can be diagonalized by a unitary matrix V as follow:

$$M_\nu = V M V^T, \tag{18}$$

where the diagonal neutrino mass matrix $M = diag(m_1, m_2, m_3)$.

If we put V is the modified neutrino mixing matrix in Eq. (13), then Eq. (18) gives the neutrino mass matrix:

$$M_\nu = \begin{pmatrix} A & B & C \\ B & D & E \\ C & E & F \end{pmatrix} = \begin{pmatrix} (M_\nu)_{11} & (M_\nu)_{12} & (M_\nu)_{13} \\ (M_\nu)_{21} & (M_\nu)_{22} & (M_\nu)_{23} \\ (M_\nu)_{31} & (M_\nu)_{32} & (M_\nu)_{33} \end{pmatrix}, \tag{19}$$

where:

$$(M_\nu)_{11} = \frac{2m_1}{3} + \frac{m_2}{3}c_y^2 + \frac{m_3}{3}s_y^2, \tag{20}$$

$$(M_\nu)_{12} = (M_\nu)_{21} = -\frac{m_1}{3} + m_2\left(\frac{1}{3}c_y^2 - \frac{\sqrt{6}}{6}c_y s_y\right) + m_3\left(\frac{1}{3}s_y^2 + \frac{\sqrt{6}}{6}s_y c_y\right), \tag{21}$$

$$(M_\nu)_{13} = (M_\nu)_{31} = -\frac{m_1}{3} + m_2\left(\frac{1}{3}c_y^2 + \frac{\sqrt{6}}{6}c_y s_y\right) + m_3\left(\frac{1}{3}s_y^2 - \frac{\sqrt{6}}{6}s_y c_y\right), \tag{22}$$

$$(M_\nu)_{22} = \frac{m_1}{6} + m_2\left(\frac{\sqrt{3}}{3}c_y - \frac{\sqrt{2}}{2}s_y\right)^2 + m_3\left(\frac{\sqrt{3}}{3}s_y + \frac{\sqrt{2}}{2}c_y\right)^2, \tag{23}$$

$$(M_\nu)_{23} = (M_\nu)_{32} = \frac{m_1}{6} + m_2\left(\frac{1}{3}c_y^2 - \frac{1}{2}s_y^2\right) + m_3\left(\frac{1}{3}s_y^2 - \frac{1}{2}c_y^2\right), \tag{24}$$

$$(M_\nu)_{33} = \frac{m_1}{6} + m_2\left(\frac{\sqrt{3}}{3}c_y + \frac{\sqrt{2}}{2}s_y\right)^2 + m_3\left(\frac{\sqrt{3}}{3}s_y - \frac{\sqrt{2}}{2}c_y\right)^2. \tag{25}$$

To simplify the problem such that we can determine the neutrino masses, which can correctly predict the neutrino mass spectrum, we impose texture zero into neutrino mass matrix in Eq. (19). Texture zero of neutrino mass matrix indicates the existence of additional symmetries beyond the Standard Model Particle Physics [23, 24]. By imposing some possibilities texture zero into Eq. (19), we then find that only one texture zero: $(M_\nu)_{11} = (M_\nu)_{13} = 0$ can correctly predict the nuetrino mass spectrum. From this texture zero pattern, we have:

$$m_2 = -1.400444385m_1,\ m_3 = -12.00741191m_1, \tag{26}$$

that implies that the neutrino mass hierarchy is normal hierarchy: $|m_1| < |m_2| < |m_3|$.

If we use the experimental value of the solar neutrino squared-mass difference (Δm^2_{21}) in Eq. (5) to determine the neutrino masses in Eq. (26), then we have:

$$m_1 = 0.00888595 \text{ eV},\ m_2 = 0.01244428 \text{ eV},\ m_3 = 0.10669729 \text{ eV}. \tag{27}$$

The obtained neutrino masses in Eq. (27) cannot give correctly the squared-mass difference for atmospheric neutrino (Δm^2_{32}) in Eq. (6). Conversely, if we use the experimental value of Δm^2_{32} in Eq. (6) to determine the value of neutrino masses in Eq. (26), then the obtained neutrino masses cannot correctly predict the squared-mass difference for solar neutrino in Eq. (5).

4. Conclusion

By introducing a simple perturbation matrix into tribimaximal mixing matrix, we then have the modified tribimaximal neutrino mixing matrix that can give nonzero $\theta_{13} = 7.89^o$ which is in agreement with the present experimental results. The neutrino mass matrix from the modified tribimaximal neutrino mixing matrix with two zeros texture predict the neutrino mass spectrum in normal hierarchy: $|m_1| < |m_2| < |m_3|$. If we use the solar neutrino squared-mass difference to determine the values of neutrino masses, then we cannot have the correct value for the atmospheric squared-mass difference. Conversely, if we use the experimental valeu of the squared-mass difference to determine the neutrino masses, then we cannot have the correct value for the solar neutrino squared-mass difference.

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The Hadronic Decay Ratios of $\eta' \to 5\pi$ at NLO in χPT

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Abstract: The hadronic decays $\eta'(958) \to 2\pi\eta$ and $\eta \to 3\pi$ give unique scientific opportunity to study symmetries in nature and to provide experimental verification of the sophisticated theoretical predictions and to reveal the driving mechanism of the decays as well as violating of isospin symmetry. We give the next-to-leading order(NLO) results in Chiral Perturbation Theory(χPT) for the ratio of different $\eta'(958) \to 5\pi$ decay rates. The hadronic decay modes are also discussed and in particular some interesting features of the η' decay are presented. The scenario we have considered shows reasonable agreement with the decay processes observed so far.

Keywords: Chiral Perturbation Theory; Hadronic Decay ; Quark Models; Mesons
PACS (2010): 12.39.Fe; 13.25.Jx?; 14.40.Be; 13.25.-k

1. Introduction

When the $\eta\prime$ discovered in 1964, there has been considerable interest in its decay both theoretically and experimentally because of its special role in low-energy scale Quantum Chromodynamics (QCD) theory. Its main decay modes, including hadronic and radiative decays, have been well measured but study of its anomalous decays is still an open field. Study of $\eta\prime$ decays provides important tests of Chiral Perturbation Theory(χPT) and other models of strong interactions, allows measurement of the EM transition form factors and tests of discrete symmetries such as C, P, CP and physics beyond the Standard model. An appropriate theoretical framework to investigate hadronic physics is provided by modelindependent calculations χPT that can be applied for strong interactions in hadronic reactions and decays[1-11]. χPT, the low energy effective theory of QCD with N_f light quark flavors, could posses different behavior for N_f=2 and 3. At small distances the

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gauge coupling decreases logarithmically, and the dynamics is successfully described by χPT close to threshold, however, for the energy range where only the massless Goldstone bosons participate [12].

$\eta\prime$ has been the center of attention in many theoretical and experimental works over the recent decades. The inclusion of the $\eta\prime$ spoils the conventional chiral counting scheme, since its mass does not vanish in the chiral limit so that higher loops will still contribute to lower chiral orders. The $\eta\prime$ is a particle that decay strongly but all its decays is suppressed. $\eta\prime$ decays offers an introduction to some of the basic strong interaction issues to study symmetries and symmetry breaking patterns.

There are many study by examining hadron decays $\eta' \to \pi\pi\eta$ [1-4]. The energy dependence of hadronic $\eta' \to \pi\pi\eta$ decay is discussed in the χPT framework in Ref. [4]. The η' can be decayed (through both of its charged and neutral decay channels) into 5π. These pions will not be correlated in terms of Bose-Einsten correlations. Preliminary results strongly support the mass decrease of the η' boson [13-15]. The $\eta' \to 3\pi$ decays do not conserve isospin since Bose symmetry forbids the three pions with $J^P = 0^-$ to occur in the iso-scalar state. Due to the large mass of η' light vector and scalar mesons could be produced in the decays. Importance of vector mesons is seen in radiative decay modes; $\eta' \to \rho\gamma$ and $\eta' \to \omega\gamma$. Contributions of light scalar mesons $\sigma(560)$, $f_0(980)$ and $a_0(980)$ should play a significant role in the decays into $\eta\pi\pi$ and $\pi\pi\pi$ [12] but it is not as apparent due to the width of the mesons and the off-shell behavior. Description of vector and scalar resonances is beyond the scope of the standard χPT but there is continuous progress in the theoretical treatment.

In the present work, we calculate the $\eta'(958) \to 5\pi$ decay rates using χPT. Within the framework of χPT, electromagnetic correction can also be described by a series of effective operators of increasing power in momentum or masses of the mesons. This will also allow us to obtain more realistic predictions for the decay $\eta\prime$. Furthermore, we first present the results which we obtain by relying purely on the numbers quoted by the Particle Data Group(PDG) in different years [16, 17, 18].

The paper is organized as follows. In next section, we give a brief description of theoretical frame work of this work. The χPT Lagrangian, definition of the different levels of Lagrangian, the scattering amplitudes and decay rates and other parameters are represented in this section. We are devoted to comparison of our results with the corresponding other experimental and theoretical results, in section III. Summary and conclusions follow in Section IV.

2. Theoretical framework

We introduced the relevant chiral Lagrangian for the calculation of octet meson masses in the three-flavor sector of u, d and s quarks, up to NLO. In this formalism, the most general chiral Lagrangian is Lorentz invariant, symmetric under chiral transformation and also contains the pseudo-scalar mesons. The effective Lagrangian consists of operators ,

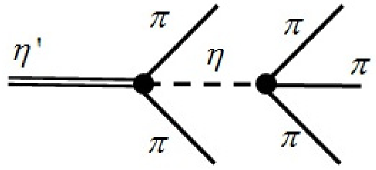

Fig. 1 The tree level diagrams contributing to $\eta' \to 5\pi$. In this figure two successive decay occurs: filled circle comes from $\mathcal{L}_2$, solid and dashed lines denote input or output pions and η particle, respectively.

with a definite number of derivatives, is given by

$$\mathcal{L}_{eff} = \mathcal{L}_2 + \mathcal{L}_4 + \ldots \,, \tag{1}$$

where $\mathcal{L}_2$, the leading order part of the Lagrangian, has the form

$$\mathcal{L}_2 = \frac{F_0^2}{4}(Tr(D_\mu U^\dagger D^\mu U) + Tr(\chi^\dagger U + \chi U^\dagger))\,, \tag{2}$$

and $D_\mu U = \partial_\mu U + \{A_\mu, U\} + [V_\mu, U]$, where A_μ and V_μ are the axial and vector currents, respectively. The $\mathcal{L}_2$ term of lagrangian contains the $SU(3)$ chiral limit of the Goldstone boson decay constant F_0 and the external pseudo-scalar sources that are conventionally combined into $\chi = 2B_0 M$ and accounts for explicit chiral symmetry breaking. B_0 is equal to $-\frac{\langle 0|\bar{q}q|0\rangle}{3F_0^2}$ and is related to the chiral quark condensate with $M = diag(m_u, m_d, m_s)$. The U matrix, including of the Goldstone boson fields and singlet meson, has the form

$$U = \exp(i\frac{\phi}{F_0})\,, \quad \phi = \begin{pmatrix} \pi^0 + \frac{1}{\sqrt{3}}\eta_8 + \frac{\sqrt{2}}{\sqrt{3}}\eta_0 & \sqrt{2}\pi^+ & \sqrt{2}K^+ \\ \sqrt{2}\pi^- & -\pi^0 + \frac{1}{\sqrt{3}}\eta_8 + \frac{\sqrt{2}}{\sqrt{3}}\eta_0 & \sqrt{2}K^0 \\ \sqrt{2}K^- & \sqrt{2}\,\overline{K^0} & \frac{-2}{\sqrt{3}}\eta_8 + \frac{\sqrt{2}}{\sqrt{3}}\eta_0 \end{pmatrix}. \tag{3}$$

where, the η_8, η_0 are related to the mass eigenstates of η, η' [1]

$$\eta_8 = \epsilon\pi^0 + \eta - \upsilon\eta', \quad \eta_0 = 3\epsilon\upsilon\pi^0 + \upsilon\eta + \eta'. \tag{4}$$

and ϵ and υ are dependent to the difference mass of u, d and s quarks(for more details see Ref. [1]).

The next-to-leading order($\mathcal{O}(p^4)$) part of the Lagrangian is given by

$$\begin{aligned} \mathcal{L}_4 = {} & L_1 Tr[D_\mu U D^\mu U^\dagger]^2 + L_2 Tr[D_\mu U D_\nu U^\dagger] Tr[D^\mu U D^\nu U^\dagger] \\ & + L_3 Tr[D_\mu U D^\mu U^\dagger D_\nu U D^\nu U^\dagger] + L_4 Tr[D_\mu U D^\mu U^\dagger] Tr(\chi U^\dagger + U\chi^\dagger) \\ & + L_5 Tr[D_\mu U D^\mu U^\dagger(\chi U^\dagger + U\chi^\dagger)] + L_6[Tr(\chi U^\dagger + U\chi^\dagger)]^2 \\ & + L_7[Tr(\chi U^\dagger - U\chi^\dagger)]^2 + L_8 Tr(\chi U^\dagger \chi U^\dagger + U\chi^\dagger U\chi^\dagger) \\ & - iL_9 Tr[f^R_{\mu\nu} D^\mu U D^\nu U^\dagger + f^L_{\mu\nu} D^\mu U^\dagger D^\nu U] + L_{10} Tr[U f^L_{\mu\nu} U^\dagger f^{R\mu\nu}]\,. \end{aligned} \tag{5}$$

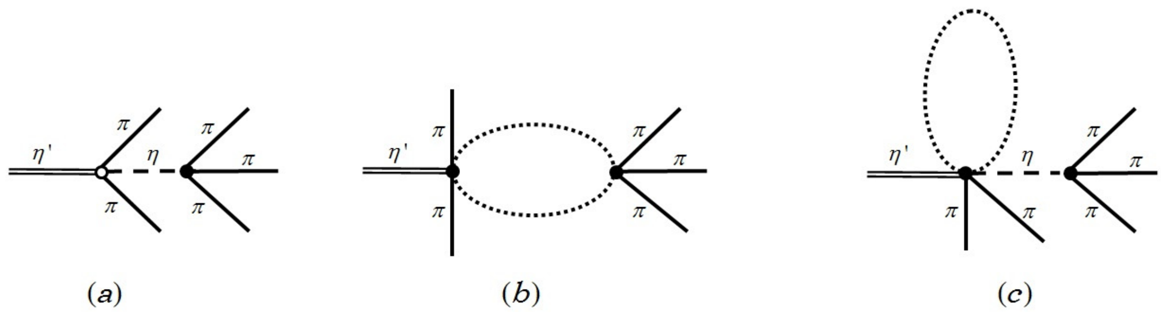

Fig. 2 Next-to-leading order diagrams contributing to $\eta' \to 5\pi$, open circle a vertex from $\mathcal{L}_4$ and other notations are similar to fig. 1.

where $f^R_{\mu\nu}$, $f^L_{\mu\nu}$ are external field strength tensors defined via

$$f^{R,L}_{\mu\nu} = \partial_\mu f^{R,L}_\nu - \partial_\nu f^{R,L}_\mu - i[f^{R,L}_\mu, f^{R,L}_\nu], \quad f^{R,L}_\mu = V_\mu \pm A_\mu. \tag{6}$$

It is noted that, the last line of the $\mathcal{L}_4$ lagrangian only contains the external fields and is not significant for low-energy physics.

The coupling constants $L_i (i = 0, ..., 12)$ are not specified from chiral symmetry. L_i's are measurable quantities and can be determined phenomenological. The relevant experiment of these parameters are obtained by appending to these values the divergent one-loop contributions

$$L_i = L_i^r + \frac{\Gamma_i}{32\pi^2} R, \quad R = \frac{2}{n-4} - [\ln(4\pi) - \gamma_E + 1] \,, \tag{7}$$

where $\gamma_E = 0.5772...$ is Euler's constant and n implication the number of space-time dimensions. The renormalization of amplitude at this order can be done when the Γ_i s are found in such a way that the divergences of the one loop Green functions divergences of the tree level diagrams of $\mathcal{L}_4$ vanished. Gasser and Leutwyler applied the background field method for calculation of the divergences at NLO [15].

The basic features of the $\eta\prime$ meson from the most recent issue of the Particle Data Group report [18] are summarized in table 1.

Here, we explain how to compute the decay amplitudes for both the charged and the neutral channel of $\eta\prime$ decays at next-to-leading chiral order p^4 considering isospin breaking up to. When the η' goes into an η and 2π, and then η decays into 3π($\eta' \to 2\pi\,\eta \to 5\pi$), there are four different CP-conserving decays

$$\begin{aligned}
&\eta'(k) \to \pi^0(p_1) + \pi^0(p_2) + \eta(p_3) \to \pi^0(p_1) + \pi^0(p_2) + \pi^0(p_4) + \pi^0(p_5) + \pi^0(p_6), \quad [A_{00000}] \,, \\
&\eta'(k) \to \pi^+(p_1) + \pi^-(p_2) + \eta(p_3) \to \pi^+(p_1) + \pi^-(p_2) + \pi^+(p_4) + \pi^-(p_5) + \pi^0(p_6), [A_{+-+-0}] \,, \\
&\eta'(k) \to \pi^+(p_1) + \pi^-(p_2) + \eta(p_3) \to \pi^+(p_1) + \pi^-(p_2) + \pi^0(p_4) + \pi^0(p_5) + \pi^0(p_6), \quad [A_{+-000}] \,, \\
&\eta'(k) \to \pi^0(p_1) + \pi^0(p_2) + \eta(p_3) \to \pi^0(p_1) + \pi^0(p_2) + \pi^+(p_4) + \pi^-(p_5) + \pi^0(p_6), \quad [A_{00+-0}] \,.
\end{aligned} \tag{8}$$

Table 1 Main properties of the $\eta\prime$ meson [18].

$M_{\eta'} = 957.793 \pm 0.06$ MeV	Avrage
$\Gamma_{\eta'} = 0.230 \pm 0.021$ MeV	Avrage
$\Gamma_{\eta'} = 0.199 \pm 0.009$ MeV	Fit
$\eta' \to \pi^+\pi^-\eta$	(43.4 $\pm$ 0.7)%
$\eta' \to \rho^0\gamma$	(29.3 $\pm$ 0.6)%
$\eta' \to \pi^0\pi^0\eta$	(21.6 $\pm$ 0.8)%
$\eta' \to \omega\gamma$	(2.75$\pm$0.22)%
$\eta' \to \gamma\gamma$	(2.18$\pm$0.08)%
$\eta' \to \pi^+\pi^-\pi^0$	(3.6$\pm$1.0)$\times10^{-3}$%
$\eta' \to \pi^+\pi^-\pi^+\pi^-\pi^0$	($<$ 1.9$\times10^{-3}$)%

Where the p and A labels are indicated the four-momentum defined for each particle and the symbol used for the each amplitude, respectively. The neutral decay is described at leading order by the diagram shown in fig. 1. The lowest-order amplitude contains neither derivatives nor electromagnetic terms. The neutral LO amplitude also has an overall factor of $m_d - m_u$, but is just a constant. Fig. 2 is also shown the contributing Feynman diagrams up to next-to-leading chiral order p^4 for $\eta\prime \to 5\pi$ decay.

The kinematics is also normally treated by using

$$\begin{aligned} s_1 &= (k-p_1)^2 \,, \quad s_2 = (k-p_2)^2 \,, \quad s_3 = (k-p_3)^2 \,, \\ s_1' &= (p_3-p_4)^2 \,, \quad s_2' = (p_3-p_5)^2 \,, \quad s_3' = (p_3-p_6)^2 \,. \end{aligned} \tag{9}$$

where $p_i^2 = m_i^2$ and $k^2 = m_{\eta'}^2$. The A_{00+-0}, A_{+-000} and A_{+-+-0} amplitudes are symmetric under the interchange of the first and second two pions, because of CP or Bose-symmetry. The $[A_{00000}]$ amplitude is also obviously symmetric under the interchange of all the final state particles.

In terms of the contributions defined before and by applying the replacement rules given above the decay amplitudes finally can be written in terms of single variable functions $M_i(s)$ and $M_i'(s)$(see for more details Ref. [19])

$$\begin{aligned} A_{00000}(s_1,s_2,s_3) &= M_0(s_1) + M_0(s_2) + M_0'(s_3)\,, \\ A_{+-+-0}(s_1,s_2,s_3) &= M_1'(s_3) + M_2(s_1) + M_2(s_2) + (s_2-s_3)M_3(s_1) + (s_1-s_3)M_3(s_2)\,, \\ A_{+-000}(s_1,s_2,s_3) &= M_4'(s_3) + M_5(s_2) + M_5(s_2) + (s_2-s_3)M_6(s_1) + (s_1-s_3)M_6(s_2)\,, \\ A_{00+-0}(s_1,s_2,s_3) &= M_7'(s_3) + M_8(s_1) + M_8(s_2) + (s_2-s_3)M_9(s_1) + (s_1-s_3)M_9(s_2)\,. \end{aligned} \tag{10}$$

The functions $M_i(s)$ and $M_i'(s)$ are not unique. Note that in case of the neutral decay the whole kinematical dependence is contained in the scattering contribution, which takes the

Table 2 Theoretical and measurement estimate of the decay widths ratios up to NLO.

	$\frac{\Gamma(\eta'\to\pi^+\pi^-\pi^+\pi^-\pi^0)}{\Gamma(\eta'\to\pi^+\pi^-\pi^0\pi^0\pi^0)}$	$\frac{\Gamma(\eta'\to\pi^0\pi^0\pi^0\pi^0\pi^0)}{\Gamma(\eta'\to\pi^+\pi^-\pi^+\pi^-\pi^0)}$	$\frac{\Gamma(\eta'\to\pi^0\pi^0\pi^0\pi^0\pi^0)}{\Gamma(\eta'\to\pi^+\pi^-\pi^0\pi^0\pi^0)}$
Ref [16]	0.402 ± 0.007	0.528 ± 0.005	0.212 ± 0.007
Ref [17]	0.399 ± 0.007	0.527 ± 0.004	0.210 ± 0.007
Ref [18]	0.400 ± 0.003	0.560 ± 0.002	0.224 ± 0.004
This work	0.479	0.580	0.278

symmetric form where discussed in the above paragraph. We have also checked both the charged and the neutral amplitude in several ways. They are finite and renormalization-scale independent.

The charged and neutral total decay widths can be calculated

$$\Gamma_{c(n)} = \frac{S_{c(n)}}{256\,\pi^3\,m_{\eta'}^3}\int |A(s_i)|^2\; ds_i\;. \tag{11}$$

where $S_{c(n)}$ is denote the symmetry factor of the charged(neutral)decay.

3. Results and discussion

We are focus to the ratio between scattering amplitudes and decay widths. Thus, we introduce some new parameters. These parameters are the ratio between scattering amplitudes in deferent decay modes

$$\begin{aligned} r_1 &= \left|\frac{A_{\eta'\to\pi^0\pi^0\pi^0\pi^0\pi^0}}{A_{\eta'\to\pi^+\pi^-\pi^+\pi^-\pi^0}}\right|^2 \quad, r_2 = \left|\frac{A_{\eta'\to\pi^0\pi^0\pi^0\pi^0\pi^0}}{A_{\eta'\to\pi^+\pi^-\pi^0\pi^0\pi^0}}\right|^2, \\ r_3 &= \left|\frac{A_{\eta'\to\pi^+\pi^-\pi^+\pi^-\pi^0}}{A_{\eta'\to\pi^+\pi^-\pi^0\pi^0\pi^0}}\right|^2 \quad, r_4 = \left|\frac{A_{\eta'\to\pi^+\pi^-\pi^+\pi^-\pi^0}}{A_{\eta'\to\pi^0\pi^0\pi^+\pi^-\pi^0}}\right|^2. \end{aligned} \tag{12}$$

The ratio of the r_1 to r_4 is shown in fig. 3. By comparison of the two diagrams fig. 3(a) and fig. 3(c), one can see that the ratio of $\eta' \to \pi^+\pi^-\pi^+\pi^-\pi^0$ is nearly about two times of scattering amplitude $\eta' \to \pi^0\pi^0\pi^0\pi^0\pi^0$. On the other hand, according to eq.(12), we conclude that decay widths of $\eta' \to \pi^+\pi^-\pi^+\pi^-\pi^0$ two times of decay widths of $\eta' \to \pi^0\pi^0\pi^0\pi^0\pi^0$ which is in good agreement to the experimental results [16, 17].

In fig. 4(a), the squared amplitude ratio $r_1\,, r_2$ and $r_3\,, r_4$, are also compared. This comparison is shown that by increasing of s_2, r_2 is growing faster than r_1. For $s_2 = 0.46$, r_1 and r_2 are equal to each other which indicates that, by increasing of s_2, the scattering amplitude in the neutral channel, is reduced. By increasing s_2, r_4 is also decreased, but r_3 is increased and for $s_2 = 0.375$, r_3 and r_4 are equal to each other, this behavior is shown in fig. 4(b).

The calculated of decay widths ratios as a function as s_2 is shown in fig. 5. Table 1 is also show the theoretical and measurement estimate of the decay widths ratios, up

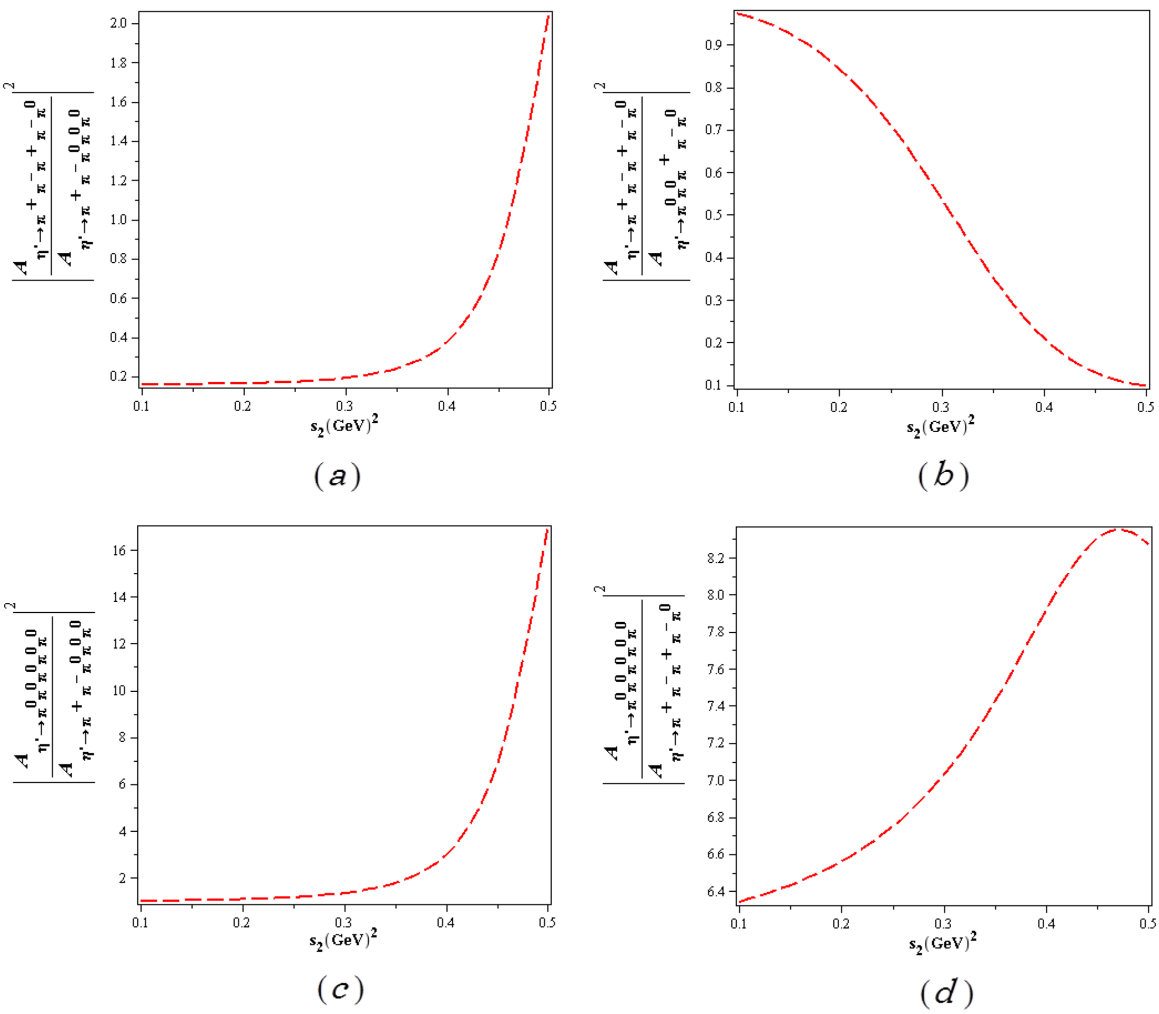

Fig. 3 Comparison of $\eta\prime \to 5\pi$ amplitudes ratio for r_1 to r_2, the horizontal axis in term s_2.

to NLO. The calculated values of the decay widths ratios and other experimental and theoretical results are compared in this table. The ratio between decay width of the charge the neutral channels, is found to be 1.72, in comparison with the its experimental value 1.90 [18]. Due to the isospin symmetric calculations of identical particles, in the isospin symmetry breaking limit, $m_u = m_d$, the decay width of charged channel $\Gamma(\eta' \to \pi^+\pi^-\pi^+\pi^-\pi^0)$ would be given by two times of the decay width of neutral channel $\Gamma(\eta' \to \pi^0\pi^0\pi^0\pi^0\pi^0)$.

4. Summary and Conclusion

Qualitatively, the scenario we have considered shows reasonable agreement with the decay processes observed so far. Further experimental input would be appreciated for the $\eta(958)$.

In this work, we have studied the hadronic η' decays in the frameworks of Chiral

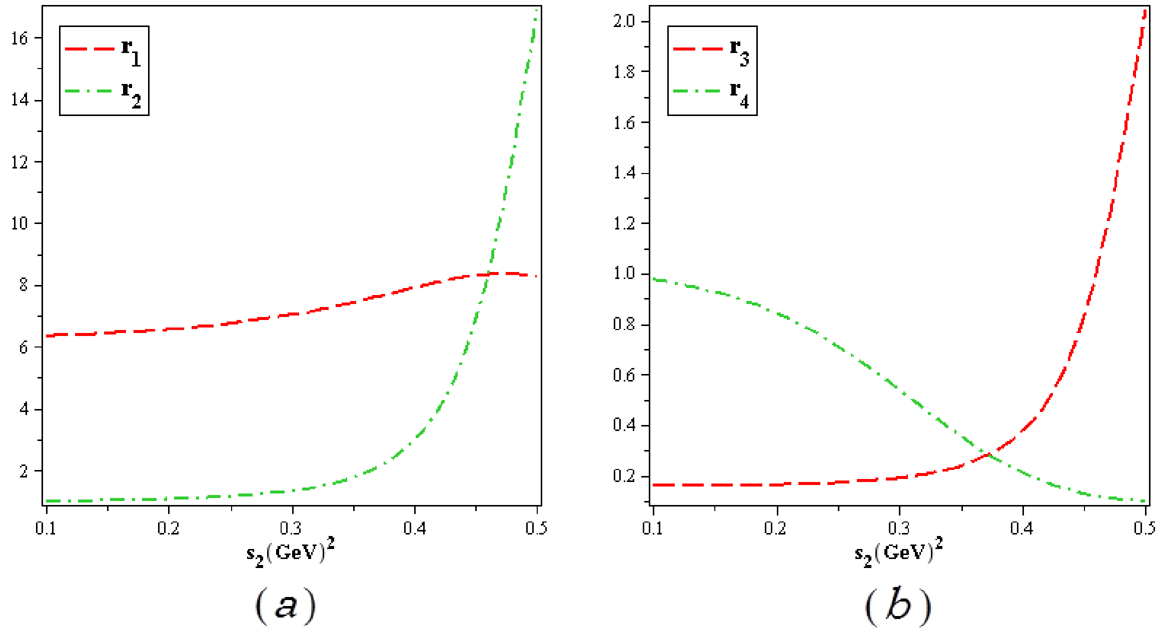

Fig. 4 Comparison of $\eta\prime \to 5\pi$ amplitudes ratio between r_1, r_2 and r_3, r_4, the horizontal axis in term s_2.

Perturbation Theory, at lowest and next-to-leading orders. We investigate the scattering amplitude η' particle decays into five particles and the scattering amplitude between the different graphing modes shown. in the end, decay widths ratios was calculated between different modes, as well as decay widths ratios between the different graphing modes is shown. We have done the calculations to NLO in χPT in the isospin limit. This work is considered the isospin breaking, the calculated results have good agreement with experimental results. To obtain more accurate results, the calculations can be extended to higher order as well.

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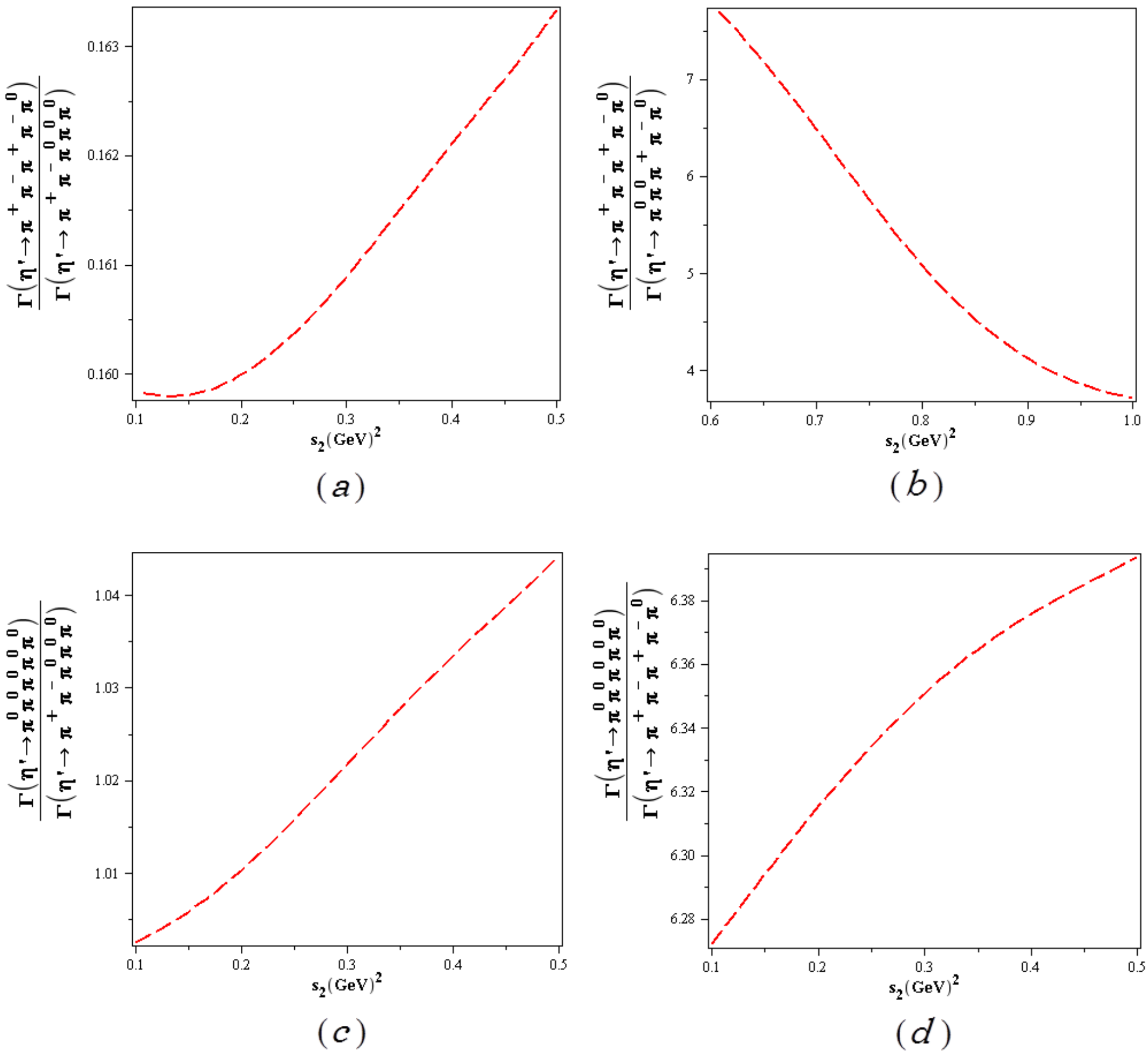

Fig. 5 Comparison of $\eta\prime \to 5\pi$ decays width ratio between different modes mentioned in this article, the horizontal axis in term s_2.

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Bound States in the Tachyon Exchange Potential

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Abstract: The Klein-Gordon field of imaginary mass is considered as a mediator of particle interaction. The static tachyon exchange potential is derived and its applied meaning is discussed. The Schrödinger equation with this potential is studied by means of variational and numerical methods. Conditions for existence of bound states are analyzed.

Keywords: Tachyon Field; Schrödinger Equation; Bound States, Critical Parameter
PACS (2010): 03.50.Kk, 03.65.Pm, 03.65.Ge

1. Introduction

The concept of faster-than-light particles – *tachyons* – is known more then a half century [1–4]. Difficulties in the quantum field theory of tachyons [5, 6] give rise to the idea that free tachyon quanta cannot exist in nature what is in agreement with experiments [7].

Nowadays, the tachyon field is rarely considered as a new kind of fundamental matter. Rather, it can serve for effective description of more or less conventional matter in unstable or metastable states. Effective tachyon fields arise within the quantum gravity [8, 9], the supersymmetric field theory [10], the string theory [11, 12] etc.

There is a complementary view of tachyons as hidden or virtual objects [13, 14]. Virtual tachyon states (whatever they are) could participate in an interaction between *bradyons* (i.e., slower-than-light particles) [15, 16]. In particular, peaks in the meson-nucleon differential cross-sections are effectively treated in [17] as tachyon resonances.

In the present paper we focus on bound states of particles interacting via tachyon field which is, in the simplest version, the Klein-Gordon field with imaginary mass. Once free tachyons are absent, the symmetric Green function of this field [6] is an appropriate

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tool for a description of the tachyon exchange interaction.

In Section 2 we derive the non-relativistic potential of the tachyon exchange interaction. An applied treatment of this potential is discussed. The Schrödinger equation with tachyon exchange potential is solved by means of both the variational approximation and the numeral integration in Section 3. Conditions for the existence of bound states are analyzed.

2. A Potential of the Tachyon Exchange Interaction

One can introduce the tachyon exchange potential in different ways. First of all, there are known in the literature several quantum-field descriptions of tachyons [5,6]. Each of them has some drawbacks or inconsistencies, e.g. Poincaré-non-invariance or/and non-unitarity of S-matrix etc. These problems are not essential when considering the tachyon exchange interaction of bradyons in the ladder approximation while free tachyons are absent. On the second way one can proceed from the classical action-at-a-distance theory of Wheeler-Feynman type [18–23] in which the electromagnetic interaction is replaced by the tachyon one. The third possibility is the partially reduced quantum field theory [24–29] which takes advantages of both above approaches. Within this framework quantized matter fields interact via mediating field (the tachyon one in present case), variables of which are eliminated from the description at the classical level.

In all these cases the tachyon exchange interaction of bradyons can be formulated in terms of the symmetric Green function of tachyon field, i.e., of the Klein-Gordon field of imaginary mass $\mathscr{M} = \mathrm{i}\,\mu$ [6], which is real and Poincare-invariant:

$$G(x) = \int \frac{\mathrm{d}^4 k}{(2\pi)^4}\, e^{ik\cdot x}\widetilde{G}(k) \qquad \text{with} \quad \widetilde{G}(k) = -\frac{\mathscr{P}}{\mu^2 + k^2}, \tag{1}$$

$x = \{x^0, \mathbf{x}\}$, $k = \{k_0, \mathbf{k}\}$ and $\mathscr{P}$ standing for the Cauchy principal value. We use the system of units in which $c = \hbar = 1$ and refer to μ as *metamass* (following [1, 2]).

Using the Green function of a mediating field one can obtain the static potential of interaction [22, 23, 27]. For the Green function (1) we have :

$$U(r) = -4\pi\alpha\int \mathrm{d}x^0\; G(x - x') = -4\pi\alpha\int \frac{\mathrm{d}^3 k}{(2\pi)^3}\mathrm{e}^{-\,\mathrm{i}\,\mathbf{k}\cdot\mathbf{r}}\tilde{G}(k_0{=}0,\, \mathbf{k}) = \alpha\frac{\cos\mu r}{r}, \tag{2}$$

where $r = |\mathbf{r}| \equiv |\mathbf{x} - \mathbf{x}'|$ is an inter-particle distance and α is a coupling constant. The static potential is sufficient for the non-relativistic description of the two-body problem.

Obviously, the tachyon exchange potential (2) is equal to the (real part of) Yukawa potential with the imaginary mass $\mathscr{M} = \mathrm{i}\,\mu$ of the mediating field. Alternatively, one can proceed from *a priori* nonrelativistic shielding potential in plasma, substituting formally the imaginary Debye radius $r_{\mathrm{D}} = \mathrm{i}\,/\mu$. Physically, this *anti-shielding* potential may occur in some metastable media such as a dielectric at negative temperature [30]. Another, astrophysical application is the gravity "dressed" by a dark matter [31].

Here we are interested in the following question: could the potential (2) yield bound states of particles, and, if yes, with which parameters?

3. The Schrödinger Equation

We suppose that the coupling constant α in (2) can be positive or negative. In both cases the potential (2) is the succession of potential wells and barriers which replace each others under the change of sign of α. Therefore within the classical (non-quantum) consideration bound states are possible with an arbitrary α (either negative or positive); they correspond to a motion of particles within one or another potential well.

The quantum two-body problem is based on the Schrödinger equation:

$$\frac{1}{2m}\Delta\Psi(\mathbf{r}) + U(r)\Psi(\mathbf{r}) = \mathscr{E}\Psi(\mathbf{r}), \tag{3}$$

where $m = m_1 m_2/(m_1 + m_2)$ is the reduced mass for particle masses m_1, m_2, and $\mathscr{E}$ is the eigen-energy of the system.

By the quantum consideration the depthes of wells or the heights of barriers of the potential (2) become important in view of the tunneling probability between them. Consequently, properties of the wave function $\Psi(\mathbf{r})$ are not obvious. Nevertheless one can expect the existence of bound states for $\alpha < 0$ and μ small when the potential (2) is close to the Coulomb one.

In order to solve the Schrödinger equation (3) we split variables into the radial and angular ones: $\Psi(\mathbf{r}) = \frac{1}{r}\psi(r)Y_\ell^\mu(\hat{r})$, where $Y_\ell^\mu(\hat{r})$ is the spherical harmonics, and $\hat{r} = \mathbf{r}/r$. It is convenient to introduce the dimensionless variables:

$$\rho = r/a_{\mathrm{B}}, \qquad \epsilon = \mathscr{E}/\mathscr{E}_{\mathrm{Ry}}, \qquad \eta = a_{\mathrm{B}}\mu, \tag{4}$$

$$\text{where} \qquad a_{\mathrm{B}} = 1/(m|\alpha|), \qquad \mathscr{E}_{\mathrm{Ry}} = m\alpha^2 \tag{5}$$

are analogs of the Bohr radius and the Rydberg constant. Then the equation for the radial wave function $\psi(\rho)$ takes the form:

$$\{H^{(\mp)} - \epsilon\}\psi(\rho) = 0, \tag{6}$$

$$\text{where} \quad H^{(\mp)} \equiv \frac{1}{2}\left\{-\frac{\mathrm{d}^2}{\mathrm{d}\rho^2} + \frac{\ell(\ell+1)}{\rho^2}\right\} + u^{(\mp)}(\rho), \tag{7}$$

$$u^{(\mp)}(\rho) = \mp\frac{\cos\eta\rho}{\rho} \equiv \mp\frac{1}{2\rho}\left\{\mathrm{e}^{\,\mathrm{i}\eta\rho} + \mathrm{e}^{-\,\mathrm{i}\eta\rho}\right\}. \tag{8}$$

The operators $H^{(\mp)}$ and $u^{(\mp)}$ correspond to the cases $\alpha = \mp|\alpha| \lessgtr 0$ which we will conventionally refer to as the "attraction" and "repulsion" ones. We note that these terms are rather provisional and they do not reflect actual properties of the interaction.

The Variational Solution

There is unknown the exact solution of the Schrödinger equation with the potential (8) which is the superposition of Yukawa potentials with imaginary exponents. We expect that the calculation of the ground state energy by means of the variational Ritz method may appear efficient, by analogy to the case of ordinary Yukawa potential [32].

Since the potential (8) is close to the Coulomb potential at $\rho \ll 1/\eta$, we choose as a trial wave function the following simple one:

$$\tilde{\psi}_{\ell 0} \equiv \sqrt{k}\psi_{\ell 0}(k\rho); \tag{9}$$

here k is the variational (scaling) parameter, and the function $\psi_{\ell 0}(\rho)$ describes the ground state ($n_r = 0$) of the radial Hamiltonian $H^{(-)}_{\eta=0}$ (7)-(8) of the Coulomb problem [33]. The function (9) is normalized and properly behaved at $\rho \to 0$.

Using the integration techniques with hypergeometric functions [33] we obtain the following expression for the average energy on the variational state (9):

$$\langle\epsilon\rangle_\varkappa \equiv \left\langle -\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}\rho^2} + \frac{\ell(\ell+1)}{2\rho^2} + u(\rho)\right\rangle_\varkappa = \frac{\eta^2}{8\varkappa^2} - \frac{\eta}{4n\varkappa}f(\varkappa), \tag{10}$$

$$\text{where} \quad f(\varkappa) = \frac{1}{(1+\mathrm{i}\,\varkappa)^{2n}} + \frac{1}{(1-\mathrm{i}\,\varkappa)^{2n}}, \tag{11}$$

$n = \ell + 1$, and the new variational parameter $\varkappa = n\eta/(2k)$ (instead of k) is used.

The minimum condition $\frac{\partial}{\partial\varkappa}\langle\epsilon\rangle_\varkappa = 0$ for the function (10) yields the equality:

$$\eta = \frac{\varkappa}{n}\{f(\varkappa) - \varkappa f'(\varkappa)\} \tag{12}$$

which, together with (10), determines the ground state energy as a parametric function (in terms of the parameter $\varkappa$) of the dimensionless metamass η.

Let us consider the ground s-state by putting $n = 1$. It follows from the equations (10)-(12) that $\langle\epsilon\rangle \le 0$ while $\varkappa \in [0, \sqrt{2}-1]$. Hereby the energy $\langle\epsilon\rangle \in [-\frac{1}{2}, 0]$ grows up monotonously together with the dimensionless metamass $\eta \in [0, 1]$. Since $\langle\epsilon\rangle$ exceeds the true values of energy, the bound state certainly exists for $\eta \in [0, 1]$. Thus we can estimate the critical value η_c of metamass (at which bound states disappear) from below as $\eta_\mathrm{c} \ge 1$.

An analytical method was developed by Sedov [34] for determining the critical screening parameter for the Yukawa potential $v(\rho) = -\exp(-\eta\rho)/\rho$. It was then applied to the related exponential cosine screened Coulomb (ECSC) potential [35]. The method leads to the equation:

$$1 - I_1/\eta_\mathrm{c} + I_2/\eta_\mathrm{c}^2 - I_3/\eta_\mathrm{c}^3 + I_4/\eta_\mathrm{c}^4 - \cdots = 0, \tag{13}$$

where the coefficients I_n ($n = 1, 2, \ldots$) can be expressed via multiple quadratures. In the present case (8) a direct application of the method fails since a part of quadratures diverges. We consider, instead of (8), the extended ECSC potential $w(\rho) = \mathrm{e}^{-\xi\rho}u^{(\mp)}(\rho)$, $\xi > 0$, and calculate asymptotic values of quadratures I_n at $\xi \to 0$. We obtain $I_1 \sim \pm 2\xi$, $I_2 \sim 2\ln\xi$, $I_3 = \mp\pi$ and $I_4 \sim 2\ln^2\xi$ but an evaluation of higher-order coefficients is an overpowering task. In 2nd- and 3rd-order approximations the equation (13) possesses the real solution $\eta_\mathrm{c} \sim \sqrt{-2\ln\xi} \to \infty$ but in 4th order such the infinite solution turns into a complex number and thus it cannot be a critical value. Nothing is known about an exact solution or higher-order solutions.

The Numeral Solution

In principle, the true values of energy can be determined with an arbitrary precision by numeral integrating the Schrödinger equation (6). Figure 1a shows the dependency $\epsilon(\eta)$ which is obtained in both the variational and numeral ways, for the Hamiltonian $H^{(-)}$, i. e. for the "attraction" case $\alpha < 0$. It is obvious the variational approximation is satisfactory all over the segment $\eta \in [0, 1[$, except for $\eta \gtrsim 1$ where it is not correct. On the other hand, numeral integration yields bound states up to $\eta \approx 2.5$. Around this value the binding energy becomes negligibly small: $|\epsilon|_{\eta > 2.5} < 10^{-9}$, and the technical difficulties of the numeral integration grow up quickly. Thus, in such a way we cannot determine the critical value of metamass η_c (if finite) at which bound states cease.

It is obvious from Figure 1b that the dependency of $1/\ln|\epsilon|$ on η tends to the asymptote with the abscissa intercept about 3. Thus bound states credibly are absent for $\eta > \eta_c \approx 3$. Of course, this assumption requires a rigorous proof which we failed to provide. In practice, an exact value of η_c is not crucially important since states with negligibly small binding energy may be considered as unbound.

The "Repulsion" Case

The variational approximation of bound states for the "repulsion" case $\alpha > 0$ can be obtained from the equations (10)-(12) as well if one puts formally $\eta < 0$ and considers that $|\eta|$ is the dimensionless metamass (instead of η). In this case $\langle\epsilon\rangle \leq 0$ for $\eta \in [0, -1]$. Thus the existence of bound states in the "repulsion" case has been proven too in despite that the used variational ansatz (9) appears extremely crude. Numerical results in figure 2a show that bound states exist for $0 < |\eta| \leq 2.7$ and probably do not exist for $|\eta| \geq 3$ (similarly to the case $\alpha < 0$); see figure 2b. The maximum of the binding energy $|\epsilon|_{\max} \approx 0.0764$ (reached at $|\eta| \approx 3/4$) is almost to the one degree smaller than one in the case $\alpha < 0$. This ability of the tachyon interaction to bind particles in both cases of negative and positive coupling constant (but with different intensions) is distinctive.

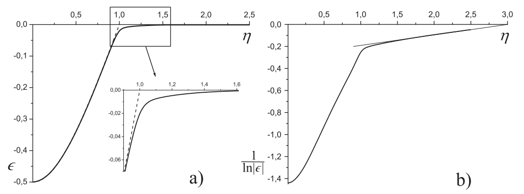

Figure 1 a) the dependency of the ground state energy on the metamass of the tachyon mediator of interaction (in the dimensionless variables ϵ, η) for the "attraction" case $\alpha < 0$: solid line – numerical results, dashed line – variational approximation; b) asymptotical behavior of the energy for $1 < \eta < 3$.

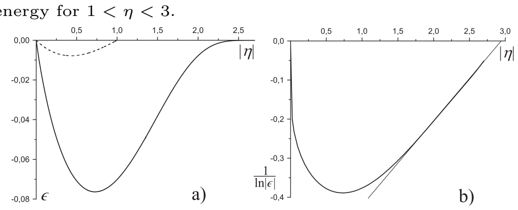

Figure 2 The same as in Figure 1, but for the "repulsion" case $\alpha > 0$.

4. Conclusions

In the two-particle problem with the tachyon exchange interaction we limit ourselves by the non-relativistic consideration and analyze the Schrödinger equation with the anti-shielding potential, i.e., the Coulomb potential modulated by $2\pi/\mu$ periodical cosine (2).

Since the exact solution of this equation is unknown, the variational and numeral methods were applied. With the former it is proved that bound states exist at both the negative and positive coupling constant α for the metamass $\mu = \eta\, m|\alpha|$ in the range $0 < \eta < 1$. The numeral calculations indicate the existence of bound states for $1 \le \eta \le 2.5$ too. Moreover, the ground state binding energy goes down quickly by increase of the metamass such that $|\mathscr{E}/\mathscr{E}_{\rm Ry}| < 10^{-9}$ at $\eta = 2.5$. Numeral integration for $\eta > 2.5$ becomes overwhelming and do not provide reliable results. The extrapolation of the dependency of $\mathscr{E}$ on the metamass in the area $\eta > 2.5$ suggests further decrease of the binding energy up to zero at $\eta_{\rm c} \simeq 3$, or $\mu_{\rm c} \simeq 3m|\alpha|$. Thus bound states exist for $\mu < \mu_{\rm c}$ or, in terms of imaginary Debye radius, for $|r_{\rm D}| > a_{\rm B}/3$. The Sedov method [34,35] does not work in the present case, thus an exact or, at least, a precise (if finite) value of the critical parameter $\eta_{\rm c}$ is an open question.

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The Origin of Randomness in Quantum Mechanics

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Abstract: A mysterious problem of randomness in quantum mechanics is revisited. This problem being hidden in the Schrödinger equation becomes transparent in its Madelung version. It has been demonstrated that randomness in quantum mechanics has the same mathematical source as that in turbulence and chaos. Special attention is concentrated on equivalence between the Schrodinger and the Madelung equations in connection with the concept of stability in dynamics.

Keywords: Quantum Mechanics; Randomness; Determinism; Schrödinger Equation; Madelung Equations
PACS (2010): 03.65.-w; 03.65.Ca; 05.45.Mt

1. Introduction

Quantum mechanics has introduced randomness into the basic description of physics via the uncertainty principle. In the Schrödinger equation, randomness is included in the wave function. But the Schrödinger equation does not *simulate* randomness: it rather describes its evolution from the prescribed initial (random) value, and this evolution is fully deterministic. The main purpose of this work is to trace down the mathematical origin of randomness in quantum mechanics, i.e. to find or build a "bridge" between the deterministic and random states. In order to do that, we will turn to the Madelung equation, [1]. For a particle mass m in a potential F, the Madelung equation takes the following form

$$\frac{\partial \rho}{\partial t} + \nabla \bullet (\frac{\rho}{m} \nabla S) = 0 \tag{1}$$

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$$\frac{\partial S}{\partial t}+\frac{1}{2m}(\nabla S)^2+F-\frac{\hbar^2\nabla^2\sqrt{\rho}}{2m\sqrt{\rho}}=0 \tag{2}$$

Here ρ and S are the components of the wave function $\psi=\sqrt{\rho}e^{iS/\hbar}$, and $\hbar$ is the Planck constant divided by 2π. The last term in Eq. (2) is known as quantum potential. From the viewpoint of Newtonian mechanics, Eq. (1) is the equation that expresses continuity of the flow of probability density, and Eq. (2) is the Hamilton-Jacobi equation for the action S of the particle. Actually the quantum potential in Eq. (2), as a feedback from Eq. (1) to Eq. (2), represents the difference between the Newtonian and quantum mechanics, and therefore, it is solely responsible for fundamental quantum properties.

The Madelung equations (1), and (2) can be converted to the Schrödinger equations using the ansatz

$$\sqrt{\rho}=\Psi\exp(-iS/h) \tag{3}$$

where ρ and S being real function.

Reversely, Eqs. (1), and (2) can be derived from the Schrödinger equation

$$i\frac{\partial\Psi}{\partial t}+\frac{\hbar^2}{2m}\nabla^2\Psi-F\Psi=0 \tag{4}$$

using the ansatz , which is reversed to (3)

$$\Psi=\sqrt{\rho}\exp(iS/\hbar) \tag{5}$$

So there is one-to-one correspondence between the solutions of the Madelung and the Schrödinger equations. From the stochastic mechanics perspective, the transformation of nonlinear Madelung equation into the linear Schrödinger equation is just a suitable mathematical technique that provides an easy way of finding their solutions.

Before starting the analysis of the Madelung equations, we have to notice that a physical equivalence of the Schrödinger and the Madelung equations is still under discussion.

In the paper [1], T. Wallstrom claims that the description of the particle's motion as a certain "conservative" diffusion is not equivalent to quantum mechanics in spite of the fact that the Madelung "hydrodynamic" equations, which provide the description of such diffusion, can be converted to the Schroedinger equation. He pointed out that such a stochastic theory can be regarded as equivalent to conventional quantum mechanics only if they can derive from it not just the Madelung equations but also the condition that the circulation of the "probability fluid" is always quantized, which is equivalent the condition for the single-valuedness of the wave function. He claims that to recover the Schrödinger equation, one must add by hand a quantization condition, as in the old quantum theory.

However as shown in [2], the single-valuedness of the wave function required in quantum mechanics, is not an auxiliary condition but a property of all local solutions of the Schrodinger equation. Based on the one-to-one correspondence between local solutions of the Schroedinger and the Madelung equations this means that the quantization of

the circulation of the "probability fluid" is a property of all solutions of the Madelung equations.

Although we incline to support the view expressed in [2], we will not make any further comments on this discussion since our target is the ***mathematical*** rather than physical equivalence between these two forms of the quantum formalism. Instead we will make the following comments:

1. In the Schrödinger equation, values of physical observables such as energy and momentum are no longer considered as values of functions on phase space, but as eigenvalues; more precisely: as spectral values of linear ***operators*** in Hilbert space.

Unlike that, the state variables of the Madelung equations preserve their classical meaning.

2. The Schrödinger equation does ***not simulate*** randomness, but rather describe its evolution in terms of the probability density, and that description is fully deterministic.

Unlike that, as will be demonstrated in this paper, Eq. (2) of the Madelung system simulates randomness: each trajectory described by the solution of this equation occurs randomly with the probability controlled by Eq. (1).

3. Since the concept of stability is not a physical invariant being dependent upon the frame of reference, upon the metric defining the distance between basic and perturbed motions, upon the class of function in which the solution is sought, etc., the solutions of the Schrödinger and Madelung equations may have different criteria of instability.

Now let us divert our attention from the physical interpretation of these equations and consider a formal mathematical problem of solving differential equations (1) and (2) subject to some initial and boundary conditions. In order not to be bounded by the quantum scale, we will assume that $\hbar$ is not necessarily the Planck constant and it can be any positive number of a classical scale having the dimensionality of action. A particular question we will ask is the following: what happen if we simulate Eqs. (1), and (2) using, for instance, electrical circuits or optical devices, and how will deterministic initial conditions generate randomness that is supposed to be present in the solutions?

2. Search for transition from determinism to randomness

Turning to Eq. (2), we will start with some simplification assuming that $F = 0$. Rewriting Eq. (2) for the one-dimensional motion of a particle, and differentiating it with respect to x, one obtains

$$m\frac{\partial^2 x(X,t)}{\partial t^2} - \frac{\hbar^2}{2m}\frac{\partial}{\partial X}\left[\frac{1}{\sqrt{\rho(X)}}\frac{\partial^2\sqrt{\rho(X)}}{\partial X^2}\right] = 0 \tag{6}$$

where $\rho(X)$ is the probability distribution of x over its possible values X.

Without the last term, Eq. (6) would represent the second Newton's law applied to the inertial motions of infinite number of *independent* samples of a particle forming a continuum $x(X)$. The last term in Eq. (6), that is a feedback from the Liouville equation, introduces an additional "force" that depends upon the probability distribution of x over

X, and thereby, it couples motions of all possible samples$x(X)$. (It should be noticed that from the viewpoint of usual interpretation of quantum mechanics, Eq. (6) is meaningless since it describes the particle trajectories that "cannot be detected").

Let us choose the following initial conditions for the deterministic state of the system:

$$x = 0, \quad \rho = \delta(|x| \to 0), \quad \dot{\rho} = 0 \text{ at } t = 0 \tag{7}$$

We intentionally did not specify the initial velocity $\dot{x}$ expecting that the solution will comply with the uncertainty principle.

Now let us rewrite the one-dimensional version of Eqs. (1) and (2) as

$$\frac{\partial^2 \rho}{\partial t^2} + \frac{\hbar^2}{2m^2}\frac{\partial^4 \rho}{\partial X^4} + \xi = 0 \text{ at } t \to 0 \tag{8}$$

where ξ includes only lower order derivatives of ρ. For the first approximation, we ignore ξ (later that will be justified,) and solve the equation

$$\frac{\partial^2 \rho}{\partial t^2} + a^2\frac{\partial^4 \rho}{\partial X^4} = 0 \text{ at } t \to 0 \quad a^2 = \frac{\hbar^2 T^2}{2m^2 L^4} \tag{9}$$

subject to the initial conditions (7). The closed form solution to this problem is known from the theory of nonlinear waves, [3]

$$\rho = \frac{1}{\sqrt{4\pi t\frac{\hbar}{2m}}}\cos\left(\frac{x^2}{4t\frac{\hbar}{2m}} - \frac{\pi}{4}\right) \text{ at } t \to 0 \tag{10}$$

Based upon this solution, one can verify that $\xi \to 0$at $t \to 0$, and that justifies the approximation (9) (for the proofs see the sub-section 2*). It is important to remember that the solution (10) is valid only for small times, and only during this period it is supposed to be positive and normalized.

Rewriting Eq. (6) in dimensionless form

$$\ddot{x} - a^2\frac{\partial}{\partial X}\left[\frac{1}{\sqrt{\rho(X)}}\frac{\partial^2\sqrt{\rho(X)}}{\partial X^2}\right] = 0 \tag{11}$$

and substituting Eq. (10) into Eq. (11) at $\boldsymbol{X} = \boldsymbol{x}$, after Taylor series expansion, simple differentiations and appropriate approximations, one arrives at the following differential equation instead of (11).

$$\ddot{x} = c\frac{x}{t^2}, \quad c = -\frac{3}{8\pi^2 a^2} \tag{12}$$

This is the Euler equation, and it has the following solution, [4]

$$x = C_1 t^{\frac{1}{2}+s} + C_2 t^{\frac{1}{2}-s} \text{ at } 4c + 1 > 0 \tag{13}$$

$$x = C_1\sqrt{t} + C_2\sqrt{t}\ln t \text{ at } 4c + 1 = 0 \tag{14}$$

$$x = C_1\sqrt{t}\cos(s\ln t) + C_2\sqrt{t}\sin(s\ln t) \text{ at } 4c + 1 < 0 \tag{15}$$

where

$$2s = \sqrt{|4c+1|} \tag{16}$$

Thus, the qualitative structure of the solution is uniquely defined by the dimensionless constant a^2 via the constants c and s, (see Eqs. (12) and (16). But the cases (14) and (15) should be disqualified at once since they are in a conflict with the approximations used for derivation of Eq. (12), (see sub-section 2*).

Hence, we have to stay with the case (13). This gives us the limits

$$0 < |c| < 0.25, \tag{17}$$

In addition to that, we have to drop the second summand in Eq. (13) since it is in a conflict with the approximation used for derivation of Eq. (9) (see sub-section 2*). Therefore, instead of Eq. (13)) we now have

$$x = C_1 t^{\frac{1}{2}+s} \text{ at } 4c+1 > 0 \tag{18}$$

For illustration, let us evaluate the constant $c$based upon the following data:

$\hbar = 10^{-34} m^2\, kg/\sec, m = 10^{-30}\, kg, L = 2.8 \times 10^{-15}\, m, L/T = \tilde{C} = 3 \times 10\, m/\sec$

where m- mass of electron, and $\tilde{C}$ - speed of light. Then,

$c = -1.5 \times 10^{-4}$, i.e. $|c| < 0.25$

Hence, the value of c is within the limit (17). Thus, for the particular case under consideration, the solution (18) is

$$x = C_1\, t^{0.9998} \tag{19}$$

In the next sub-section, prior to analysis of the solution (18), we will present the proofs justifying the solution (10).

2*. Proofs.

1. Let us first justify the statement that $\xi \to 0$ at $t \to 0$ (see Eq. (8).

For that purpose, consider the solution(10)

$$\rho = \frac{1}{\sqrt{4\pi a t}} \cos(\frac{X^2}{4 \text{ at}} - \frac{\pi}{4}) \text{ at } t \to 0 \tag{1*}$$

As follows from the solution (18),

$$\frac{x}{t} \approx o(t^{s-1/2}) \to \infty, \frac{x^2}{t} \approx o(t^{2s}) \to 0 \text{ at } t \to 0 \text{ since } 0 < s < 1/2 \tag{2*}$$

Then, finding the derivatives from Eq. (1') yields

$$|\frac{\partial^n \rho}{\partial X^n}|/|\frac{\partial^{n-1}\rho}{\partial X^{n-1}}| \approx o(t^{-1}) \to \infty \text{ at } t \to 0 \tag{3*}$$

and that justifies the inequalities

$$|\frac{\partial^4 \rho}{\partial X^4}| >> |\frac{\partial^3 \rho}{\partial X^3}|, |\frac{\partial^2 \rho}{\partial X^2}|, |\frac{\partial \rho}{\partial X}|, \rho \tag{4*}$$

Similarly,

$$|\frac{\partial^n \rho}{\partial t^n}|/|\frac{\partial^{n-1} \rho}{\partial t^{n-1}}| \approx o(t^{-1}) \to \infty \text{ at } t \to 0 \tag{5*}$$

and that justifies the inequalities

$$|\frac{\partial^2 \rho}{\partial t^2}| >> |\frac{\partial \rho}{\partial t}|, |\frac{\partial \rho}{\partial X}|^2$$

Also as follows from the solution (18)

$$\begin{aligned} &|\tfrac{\partial S}{\partial x}| \approx o(t^{S-0.5}),\ |\tfrac{\partial^2 S}{\partial x^2}| \approx o(t^{-1}), \\ &|\tfrac{\partial S}{\partial x}|/|\tfrac{\partial^2 S}{\partial x^2}| \approx o(t^{S+0.5}) \to 0 \text{ at } t \to 0 \end{aligned} \tag{6*}$$

It should be noticed that for Eq. (13), the evaluations (6*) do not go through, and that was the reason for dropping the second summand.

Finally, the inequalities (4*), (5*) and (6*) justify the transition from Eq. (8) to Eq. (10).

2. Next let us first prove the positivity of ρ in Eq. (10) for small times. Turning to the evaluation (2*)

$\frac{x^2}{t} \approx o(t^{2s}) \to 0$ at $t \to 0$, one obtains for small times

$$\rho = \frac{1}{\sqrt{4\pi a t}} \cos(-\frac{\pi}{4}) > 0 \text{ at } t \to 0 \tag{7*}$$

In order to prove that ρ is normalized for small times, turn to Eq.(9) and integrate it over X

$$\int_{-\infty}^{\infty} \frac{\partial^2 \rho}{\partial t^2} dX + a^2 \int_{-\infty}^{\infty} \frac{\partial^4 \rho}{\partial X^4} dX = 0 \tag{8*}$$

Taking into account the initial conditions (7) and requiring that ρ and all its space derivatives vanish at infinity, one obtains

$$\frac{\partial^2}{\partial t^2} \int_{-\infty}^{\infty} \rho dX = 0 \tag{9*}$$

But as follows from the initial conditions (7)

$$\int_{-\infty}^{\infty} \rho dX = 0, \frac{\partial}{\partial t} \int_{-\infty}^{\infty} \rho dX = 0 \text{ at } t = 0 \tag{10*}$$

Combining Eqs. (9*) and (10*), one concludes that the normalization constraint is preserved during small times.

3. The solutions (13), (14) and (15) have been derived under assumption that

$$\frac{x^2}{t} \to 0 \text{ at } t \to 0 \tag{20}$$

since this assumption was exploited for expansion of ρ in Eq. (10) in Taylor series. However, in the cases (14) and (15),

$$\frac{x^2}{t} \approx o(1) \text{ at } t \to 0,$$

and that disqualify their derivation. Actually these cases require an additional analysis that is out of scope of this paper. For the same reason, Eq. (13) has been truncated to the form (18).

3. Analysis of solution

Turning to the solution (18), we notice that it satisfies the initial condition (7) i.e. $x = 0$ at $t = 0$ for *any* values of C_1 : all these solutions co-exist in a superimposed fashion; it is also consistent with the sharp initial condition for the solution (10) of the corresponding Liouvile equation (1). The solution (10) describes the simplest *irreversible* motion: it is characterized by the “beginning of time” where all the trajectories intersect (that results from the violation of the Lipchitz condition at $t = 0$, Fig.2); then the solution splits into a continuous set of random samples representing a stochastic process with the probability density ρ controlled by Eq. (10). The irreversibility of the process follows from the fact that the backward motion obtained by replacement of t with $(-t)$ in Eqs. (10) and (18) leads to imaginary values. Actually Fig. 1 illustrates a jump from determinism to a coherent state of superimposed solutions that is ***lost in solutions of the Schrödinger equation.***

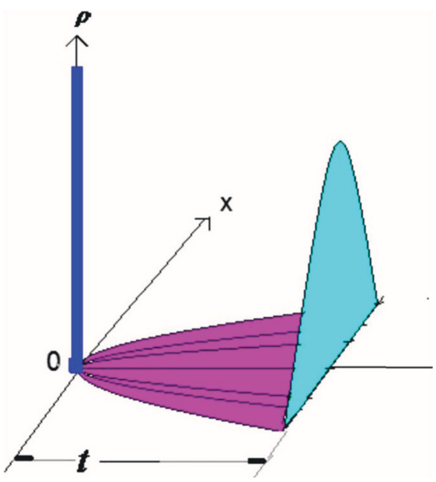

Fig. 1 Hidden statistics of transition from determinism to randomness.

Let us show that this jump is triggered by instability of the deterministic state. Indeed, turning to the solution represented by Eq. (18) with $|C_1| \leq 0.25$, we observe that

for fixed values of C_1, the solution (18) is ***unstable*** since

$$\frac{d\dot{x}}{dx} = \frac{\ddot{x}}{\dot{x}} > 0 \tag{21}$$

and therefore, an initial error always grows generating ***randomness***. Initially, at *t =0*, that growth is of *infinite rate* since the Lipchitz condition at this point is violated (such a point represents a ***terminal*** repeller,)

$$\frac{d\dot{x}}{dx} \to \infty \text{ at } t \to 0 \tag{22}$$

This means that an *infinitesimal* initial error becomes finite in a bounded time interval. That kind of instability (similar to blow-up, or Hadamard, instability) has been analyzed in [5]. Considering first Eq.(18) at fixed C_1 as a sample of the underlying stochastic process (13), and then varying C_1, one arrives at the whole ensemble of one-parametrical random solutions characterizing that process, (see Fig.2). It should be stressed again that this solution is valid only during a small initial period representing a "bridge" between deterministic and random states, and that was essential for the derivation of the solutions (18), and (10).

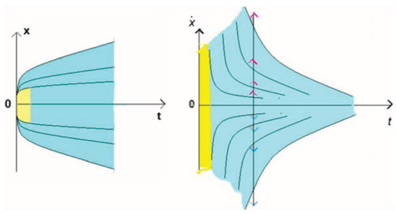

Fig. 2 Family of random trajectories and particle velocities.

Returning to the quantum interpretation of Eqs. (1) and (2), one notice that during this transitional period, the quantum postulates are preserved. Indeed, as follows from Eq. (21),

$$\dot{x} \to \infty \text{ at } t \to 0 \tag{23}$$

i.e. the initial velocity is *not defined*, (see the yellow areas in Fig. 2), and ***that confirms the uncertainty principle.*** It is interesting to note that an enforcement of the initial velocity would "blow-up" the solution (18); at the same time, the qualitative picture of the solution is not changed if the initial velocity is not enforced: the solution is composed of superposition of a family of random trajectories with the singularity (23) at the origin. Next, the solution (18) justifies the *belief* sheared by the most physicists that particle trajectories do not exist, although, to be more precise, as follows from Eq. (18), *deterministic* trajectories do not exist: each run of the solution (18) produces different trajectory that occurs with probability governed by Eq. (10). It is easily verifiable that the transition of motion from one trajectory to another is very sensitive to errors in initial conditions in the neighborhood of the deterministic state. Indeed, as follows from Eq. (18),

$$C_1 = x_0 t_0^{-(s+0.5)}, \ \frac{\partial C_1}{\partial x_0} = t_0^{-(s+0.5)} \to \infty \text{ as } t_0 \to 0 \tag{24}$$

where x_0 andt_0 are small errors in initial conditions.

Actually Eq. (18) represents a hidden statistics of the underlying Schredinger equation. As pointed out above, the cause of the randomness is non-Lipchitz instability of Eq. (18) at *t =0*. Therefore, trajectories of quantum particles have the same "status" as trajectories of classical particles in a turbulent or chaotic motion with the only difference that the "choice" of the trajectory is made only at $t_0 \to 0$. It should be emphasized again that the transition (18) is *irreversible*. However, as soon as the difference between the current probability density and its initial sharp value becomes finite, one arrives at the conventional quantum formalism described by the Schrödinger, as well as the Madelung equations. Thus, in the conventional quantum formalism, the *transition* from the classical to the quantum state has been *lost*, and that created a major obstacle to interpretation of quantum mechanics as an extension of the Newtonian mechanics. However, as demonstrated above, the quantum and classical worlds can be reconciled via the more subtle mathematical treatment of the ***same equations***. This result is generalizable to multi-dimensional case as well as to case with external potentials.

4. Comments on equivalence of Schrödinger and Madelung equations

Equivalence of Schrödinger and Madelung equations was questioned by some quantum physicists on the ground that to recover the Schrödinger equation from the Madelung equation, one must add by hand a quantization condition, as in the old quantum theory. However, this argument has been challenged by other physicists. We will not go into details of this discussion since we will be more interested in ***mathematical*** rather than physical equivalence of Schrödinger and Madelung equations. Firstly we have to notice that the Schrödinger equation is more attractive for computations due to its linearity, while the Madelung equations have a methodological advantage: they allow one to trace down the Newtonian origin of the quantum physics. Indeed, if one drops the Planck's constant, the Madelung equations degenerate into the Hamilton-Jacobi equation supplemented by the Liouville equation. However despite the fact that these two forms of the same governing equations of quantum physics can be obtained from one another (see Eqs. (3) and (5)) without a violation of any of mathematical rules, there is more significant difference between them, and this difference is associated with the concept of stability.

4.1 Stability in Physics

Any mathematical model of a continuum should be tested for three properties: existence, uniqueness and stability of its solutions. However, none of these properties are physical invariants since they depend upon a mathematical setting of the corresponding model. As an example, consider a vertical, ideally flexible filament OA with a free lower end A suspended in the gravity field at the point O, Fig. 3.

As shown in textbooks on analytical mechanics, [6], the problem of small oscillations

Fig. 3 Snap of a whip – violation of the Lipchitz condition at the free end

of a filament with respect to its vertical position is described by a unique and stable solution ***if the Lipchitz condition is enforced***. However this condition suppresses the snap at the free end that is well known from simple experiments, and therefore, it is far from physical reality. Revision of this solution was made in [7]. Omitting mathematical details, we will describe here the physical argumentation that leads to a snap at the free end.

The tension T of the filament due to gravity is the following

$$T = \gamma(L - x) \tag{25}$$

where γ and L are the specific weight and length of the filament.

4.1.0.1 Since the characteristic speed λ of a transverse wave in ideal filaments is

$$\lambda = \sqrt{\frac{T}{\rho}} \tag{26}$$

this speed vanishes at the free end

$$T|_{x=L} = 0, \quad \lambda = 0 \text{ at } x \to 0 \tag{27}$$

In other words, for small transverse displacements of the filament, the governing equation is of hyperbolic type only in the open interval that excludes the free end

$$0 \leq x < L \tag{28}$$

As shows in [7], in this ***open*** interval there exists a unique stable solution.

However, in the ***closed interval*** that includes the free end

$$0 \leq x \leq L \tag{29}$$

the solution is not unique, and there are unstable solutions since the improper integral

$$\int_0^x \frac{d\xi}{\sqrt{T(\xi)/\rho}} \tag{30}$$

converges for $x \to L$.

This result has a clear physical interpretation: suppose that an isolated transverse wave of small amplitude was generated at the point of suspension O, Fig. 3. Then the

speed of propagation of its leading front will be smaller than the speed of propagation of the trailing front because the tension decreases from the point of suspension to the free end (see Eq. (25))). Hence the length of the wave will decrease and vanish ***at*** the free end. Then according to the law of conservation of energy, the kinetic energy per unit of length will tend to infinity and produce a snap. It can be verified that the Lipchitz condition ***at*** the free end is violated

$$\frac{d\dot{x}}{dx} \to \infty \text{ at } x \to L \tag{31}$$

in the closed interval (29).

Thus it turns out that the unique stable solution exists in the class of functions satisfying the Lipchitz condition. However despite its "nice" mathematical properties, this solution is in contradiction with experiments: the cumulative effect – snap of a whip – is lost. At the same time, the removal of the Lipchitz conditions leads to non-unique unstable solutions that perfectly describe the snap of a whip.

This trivial example leads to an important conclusion: existence, uniqueness and stability of solutions of PDE describing dynamics of a continuum are ***not*** physical invariants: they are attributes of underlying ***mathematical model***. It also becomes clear that in some cases, unstable and non-unique solutions are closer to reality than a unique and stable one.

4.2 Euler/Lagrange description of fluid

In this subsection, we prepare a reader to analogy between the Schrödinger/Madelung description of quantum mechanics and the Euler/Lagrange description of fluid.

The Lagrange's description of fluid has advantage over the Euler's one in case of studies of mixing and dispersion. That includes mixing of passive scalars and the dispersion of contaminants..

The Lagrange's description of motion starts with the frame of reference that is frozen at the fluid and moves with it. Initially, at $t_0 = 0$, such frame can be represented by Cartesian axes X_0, Y_0, and Z_0.

For t >0, these axes, in general, transform into a non-orthogonal curvilinear system. Any individualized particle of the fluid with the coordinates x_0, y_0, and z_0 will have the same coordinates in the mowing frame, but different coordinates in the initial frame

$$x = x(x_0, y_0, z_0, t) \tag{32}$$

$$y = y(x_0, y_0, z_0, t) \tag{33}$$

$$z = z(x_0, y_0, z_0, t) \tag{34}$$

Eqs. (32), (33) and (34) represent the Lagrange's description of a continuum, including fluid. These equations can be obtained as the solution of the governing equations of the fluid in Euler's description

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = F_x - \frac{1}{\rho}\frac{\partial p}{\partial x} \tag{35}$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}=F_y-\frac{1}{\rho}\frac{\partial p}{\partial x} \tag{36}$$

$$\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}=F_z-\frac{1}{\rho}\frac{\partial p}{\partial x} \tag{37}$$

$$\frac{\partial}{\partial x}u+\frac{\partial}{\partial y}v+\frac{\partial}{\partial z}w=0, \tag{38}$$

presented in the Lagrange's form

$$(F_x-\frac{\partial^2 x}{\partial t^2})\frac{\partial x}{\partial x_0}+(F_y-\frac{\partial^2 y}{\partial t^2})\frac{\partial y}{\partial x_0}+(F_z-\frac{\partial^2 z}{\partial t^2})\frac{\partial z}{\partial x_0}=\frac{1}{\rho}\frac{\partial p}{\partial x_0} \tag{39}$$

$$(F_x-\frac{\partial^2 x}{\partial t^2})\frac{\partial x}{\partial y_0}+(F_y-\frac{\partial^2 y}{\partial t^2})\frac{\partial y}{\partial y_0}+(F_z-\frac{\partial^2 z}{\partial t^2})\frac{\partial z}{\partial y_0}=\frac{1}{\rho}\frac{\partial p}{\partial y_0} \tag{40}$$

$$(F_x-\frac{\partial^2 x}{\partial t^2})\frac{\partial x}{\partial z_0}+(F_y-\frac{\partial^2 y}{\partial t^2})\frac{\partial y}{\partial z_0}+(F_z-\frac{\partial^2 z}{\partial t^2})\frac{\partial z}{\partial z_0}=\frac{1}{\rho}\frac{\partial p}{\partial z_0} \tag{41}$$

$$\begin{vmatrix} \frac{\partial x}{\partial x_0} & \frac{\partial y}{\partial x_0} & \frac{\partial z}{\partial z_0} \\ \frac{\partial x}{\partial y_0} & \frac{\partial y}{\partial y_0} & \frac{\partial z}{\partial y_0} \\ \frac{\partial x}{\partial z_0} & \frac{\partial y}{\partial z_0} & \frac{\partial z}{\partial z_0} \end{vmatrix}=1 \tag{42}$$

This is a system of four PDE with respect to four unknowns : x, y, z and p as functions of $x_0, y_{0,}z_0$ and t.

Here p and $\boldsymbol{F}$ are pressure and force, respectively.

However if the Euler's equations (35) – (38) are already solved in the form

$$\begin{aligned} u &= u(x,y,z,t) \\ v &= v(x,y,z,t) \\ w &= w(x,y,z,t) \end{aligned} \tag{43}$$

then the Lagrangian description of the same motion reduces to three kinematical ODE

$$\begin{aligned} \tfrac{dx}{dt} &= u(x,y,z,t) \\ \tfrac{dy}{dt} &= v(x,y,z,t) \\ \tfrac{dz}{dt} &= w(x,y,z,t) \end{aligned} \tag{44}$$

to be solved subject to the following initial conditions

$$x=x_0, \quad y=y_0, \quad z=z_0 \text{ at } t=0 \tag{45}$$

or in the vector form

$$\frac{d\mathbf{r}}{dt}=\mathbf{v}(\mathbf{r},t) \tag{46}$$

The solution of this system can be written in the form (32) –(34).

Actually Eq. (46) is the analog of Eq. (3): it describes the transition from the Euler's to Lagrange's description of fluid in the same way as Eq. (3) describes the transition from the Schrödinger's to Madelung description of quantum mechanics. The inverse transition similar to Eq. (5) is

$$\mathbf{v}_L(\mathbf{r}_0, t) = \mathbf{v}_E[\mathbf{r}_0(\mathbf{r}, t), t] \tag{47}$$

(see Eqs. (32-34))

For further analysis we will restrict the problem to stationary laminar flows and consider the autonomous version of Eqs. (44)

$$\frac{dx}{dt} = u(x, y, z) \tag{48}$$

$$\frac{dy}{dt} = v(x, y, z) \tag{49}$$

$$\frac{dz}{dt} = w(x, y, z) \tag{50}$$

The first theoretical example of the case when the flow in the Euler's description is ***stable***, but in the Lagrange's description is ***unstable*** was introduced by V. Arnold, [8].

He considered a 3D inviscid stationary shear flow with a ***smooth*** velocity field

$$u = A \sin z + C \cos y, \qquad v = B \sin x + A \cos z, \qquad w = C \sin y + B \cos x \tag{51}$$

(see Fig.4)

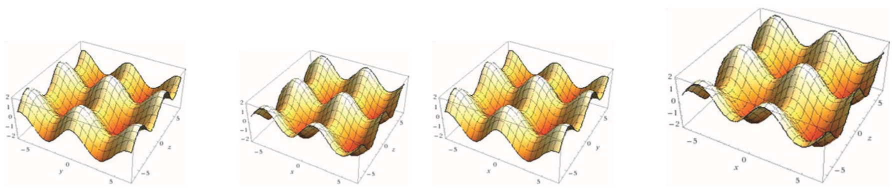

Fig. 4 Cartesian components of the velocity vector.

The solution (51) is differentiable as many times as needed, it satisfies the Euler equations (35)-(38), and it is stable. However if one tries to find the trajectories of individual particles by transition to the Lagrangian description, i.e.to Eqs. (48)-(50), he finds that after substitution the solution (51), these equations

$$\frac{dx}{dt} = A \sin z + C \cos y \tag{52}$$

$$\frac{dy}{dt} = B \sin x + A \cos z \tag{53}$$

$$\frac{dz}{dt} = C \sin y + B \cos x \tag{54}$$

are unstable, and their solution is chaotic. This means that a flow with stable stationary deterministic velocity field can have non-stationary random trajectories of individual particles. In other words, it means that this flow is stable in the Eulerian coordinates,

but is unstable in the Lagrangian coordinates. Experimentally this phenomenon can be captured if some particles of the flow are marked by different colors. This interesting and surprising phenomenon opened up a new direction in fluid mechanics known as Lagrangian turbulence.

4.3 Schrödinger/Madelung and Euler/Lagrange analogy

Actually the result presented above demonstrates the analogy between quantum mechanics and fluid mechanics. Indeed, as demonstrated in Sections 2 and 3 of this paper, the solution of the Madelung equations with deterministic initial condition (7) is unstable, and it describes the jump from the determinism to randomness. This illuminates the origin of randomness in quantum physics. However the Schrödinger equation does not have such a solution; moreover, it does not "allow" one to pose such a problem and that is why the randomness in quantum mechanics had to be postulated. So what happens with mathematical equivalence of the Schrödinger and the Madelung equations? In order to answer this question, let us turn again to the concept of stability. It should be recalled that stability is not an invariant of a physical model. It is an attribute of its mathematical description: it depends upon the frame of reference, upon the class of functions in which the motion is presented, upon the metrics of configuration space, and in particular, upon the way in which the distance between the basic and perturbed solutions is defined. One should recall that stability analysis is based upon a departure from the basic state into a perturbed state, and such departure requires an expansion of the basic space. However, Schrödinger and Madelung equations in the ***expanded*** spaces are not necessarily equivalent any more, and that explains the difference in the concept of stability of the same solution as well as the interpretation of randomness in quantum mechanics.

There is another "mystery" in quantum mechanics that can be clarified by transition to the Madelung space: a ***belief*** that a particle trajectory does not exist. Indeed, let us turn to Eq. (18). For any particular value of the arbitrary constant C_1, it presents the corresponding particle's trajectory. However as a result of the Lipchitz instability at $t = 0$, this constant is supersensitive to infinitesimal disturbances, and actually it becomes random at $t = 0$. That makes random the choice of the whole trajectory, while the randomness is controlled by Eq. (1). Actually this provides a justification for the ***belief*** that a particle can occupy any place at any time: it is due to randomness of its trajectory. However it should be emphasized that the particle makes random choice only once:

at $t = 0$. After that it stays on the chosen trajectory. Therefore in our interpretation this belief does not mean that a trajectory does not exist: it means only that the trajectory exists, but it is unstable. Based upon that, we can extract some deterministic information about the particle trajectory by posing the following question: find such a trajectory that has the highest probability to appear. The solution of this problem is straight forward: in the process of collecting statistics for the arbitrary constant C_1find such its value that has the highest frequency to appear. Then the corresponding trajectory will have the

highest probability to appear as well.

Thus, for the problem with the initial condition (7), the Schrödinger and Madelung equations are equivalent only in the open time interval

$$t > 0, \tag{55}$$

since the Schrödinger equation does not include the infinitesimal area around the singularity at

$$t = 0 \tag{56}$$

while the Madelung equation exists in the closed interval

$$t \geq 0 \tag{57}$$

But all the “machinery” of randomness emerges precisely in the area around the singularity (56). That is why the source of randomness is missed in the Schrödinger equation, and the randomness had to be postulated.

Hence although historically the Schrödinger equation was proposed first, and only after a couple of months Madelung introduced its hydrodynamic version that bears his name, strictly speaking, the foundations of quantum mechanics would be saved of many paradoxes had it be based upon the Madelung equation.

5. Summary

Reformulation of quantum mechanics using the Madelung equation allows one to clarify the origin of randomness and justify the belief that a particle can occupy any position at any time. The clarifications are based upon the blow-up instability of a deterministic state due to failure of Lipchitz condition. This property does not exist in Hilbert space formulation. It has been demonstrated that randomness in quantum mechanics has the same mathematical source as that in turbulence and chaos. Special attention is concentrated on equivalence between the Schrodinger and the Madelung equations in connection with the concept of stability in dynamics.

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Novel and Unique Expression for the Radiation Reaction Force, Relevance of Newton's Third Law and Tunneling

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Abstract: We derive the radiation reaction by taking into account that the acceleration of the charge is caused by the interaction with some heavy source particle. In the non relativistic case this leads, in contrast to the usual approach, immediately to a result which is Galilei invariant. Simple examples show that there can be small regions of extremely low velocity where the energy requirements cannot be fulfilled, and which the charged particle can only cross by quantum mechanical tunneling. We also give the relativistic generalization which appears unique. The force is a four-vector, but only if the presence of the source is taken into account as well. It contains no third derivatives of the position as the Lorentz-Abraham-Dirac equation, and consequently no run away solutions. All examples considered so far give reasonable results.

Keywords: Classical and Relativistic Field Theories ; Radiation Reaction; Larmor Formula
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1. Introduction: What has gone wrong?

"The problem of radiation reaction and the self force is the oldest unsolved mystery in physics" as stated in the review article of Hammond [1]. We refer to this article for details about history, suggested solutions, and literature.

Let us very briefly recall the standard derivation of the reaction force, the short comings in this derivation, as well as the problems arising with the result obtained in this way.

The starting point is the well known Larmor formula for the energy loss per time of

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a charged particle, accelerated by some external force:

$$\frac{dE}{dt} = -\frac{2}{3}\frac{e^2}{c^3}\dot{\mathbf{v}}^2. \tag{1}$$

One argues that this energy loss has to be compensated by the power due to a radiation reaction force $\mathbf{f}_{rad}$ which acts on the particle:

$$\mathbf{f}_{rad}\,\mathbf{v} = -\frac{2}{3}\frac{e^2}{c^3}\dot{\mathbf{v}}^2. \tag{2}$$

Subsequently one integrates this relation over some time interval and performs a partial integration:

$$\int_{t_1}^{t_2} \mathbf{f}_{rad}\,\mathbf{v}dt = -\frac{2}{3}\frac{e^2}{c^3}\int_{t_1}^{t_2} \dot{\mathbf{v}}^2 dt = \frac{2}{3}\frac{e^2}{c^3}\Big(\int_{t_1}^{t_2} \ddot{\mathbf{v}}\mathbf{v}dt - \dot{\mathbf{v}}\mathbf{v}|_{t_1}^{t_2}\Big). \tag{3}$$

One now assumes that $\dot{\mathbf{v}}\mathbf{v}$ vanishes at the end points and boldly concludes

$$\mathbf{f}_{rad} = \frac{2}{3}\frac{e^2}{c^3}\ddot{\mathbf{v}}. \tag{4}$$

Finally one can write down the relativistic generalization F^{μ}_{rad} of the radiation reaction force:

$$F^{\mu}_{rad} = \frac{2}{3}\frac{e^2}{c^3}\Big(\frac{d^2u^{\mu}}{d\tau^2} + \frac{du_{\nu}}{d\tau}\frac{du^{\nu}}{d\tau}\frac{u^{\mu}}{c^2}\Big), \tag{5}$$

with u^{μ} the velocity four vector and τ the proper time. The second term has been added in order to ensure $u_{\mu}F^{\mu}_{rad} = 0$. The corresponding equation of motion is often called the Lorentz-Abraham-Dirac (LAD-) equation [2][3][4].

Maliciously speaking one may say that the Larmor formula (1) is the very last formula in this chain which is correct. Already (2), though presented in an innumerable number of text books, must obviously be wrong - from the simple reason that it is not Galilei invariant! Under a non relativistic boost, i.e. a Galilei transformation of the form $\mathbf{x} = \mathbf{x}' + \mathbf{w}t$, forces and accelerations are invariant, while velocities change according to $\mathbf{v} = \mathbf{v}' + \mathbf{w}$. Clearly eq. (2) is not Galilei invariant.

The assumptions made after the partial integration are as well dubious and rarely ever fulfilled. The integrated terms $\dot{\mathbf{v}}\mathbf{v}|_{t_1}^{t_2}$ are again not Galilei invariant and only vanish for rather special cases. Nevertheless one appears astonished when the resulting equation, applied to situations where the assumption is definitely not fulfilled, gives unphysical run away solutions. It is also not mandatory to conclude from the equality of the integrals to the equality of the integrands. And, finally, one only gets an information on the component of the force parallel to the velocity. The resulting equation of motion is also strange because it contains the third time derivative of the position, a rather unfamiliar feature. For the initial value problem one has to prescribe not only position and velocity, but also the acceleration.

In the present work we will approach and solve the problem in a direct way. An essential point is to give up the unphysical concept of an external field which is implicitly

present in the usual procedure. Instead we assume that the "external" force arises through the interaction with a second (very heavy) source particle. We work directly with the equations for momentum and energy, without performing partial integrations or other manipulations, or needing any assumptions about the internal structure of the charge.

In the non relativistic case treated in sect. 2 our procedure corrects immediately the wrong equation (2), introduces the difference between the velocities of particle and source, and leads to a result which is Galilei invariant. No third derivatives $\dddot{\mathbf{x}}$ appear, one has the standard initial value problem of prescribing position and momentum, no run away solutions show up. For applications we assume, of course, that the mass of the source particle is very large, so it becomes essentially a static source. We apply our equation to three simple problems: Constant external force, harmonic oscillator, and circular motion. Some of the solutions show a remarkable property, at first sight a serious defect, but indeed an inevitable phenomenon which can be well understood and will be explained in detail later. In some (very tiny) regions there is no solution because the energy requirements cannot be fulfilled. The mathematics reacts by producing a complex solution in these regions. This indicates that we have to leave the field of classical physics here and must enter the world of quantum mechanics. The charged particle has to tunnel through a classically forbidden region.

In sect. 3 we generalize to relativistic motion. We construct a four-force which we consider as unique. It has all the properties which are required. The manifestly covariant form, as was to be expected from the non relativistic case, can only be obtained if one considers the source particle as well. We apply our equation to the linear relativistic motion in a constant electric field. This leads to a reasonable result.

Sect. 4 summarizes the properties of the radiation reaction force obtained here.

2. Non relativistic equation

We abandon the unphysical concept of an external force as reason for the acceleration and replace it by the force created by a large mass M situated at $\mathbf{X}$, which interacts with the considered particle of mass m and charge e via a potential $\varphi(\mathbf{x}-\mathbf{X})$. The velocities of the particle and the source are denoted by $\mathbf{v}$ and $\mathbf{V}$. The source particle M has no charge for simplicity.

For non relativistic motion one can ignore the radiation of momentum, there is only radiation of energy. Momentum conservation then gives $m\dot{\mathbf{v}}+M\dot{\mathbf{V}}=0$, while the energy loss per time becomes

$$\begin{aligned}&\frac{d}{dt}\Big(\frac{m}{2}v^2+\frac{M}{2}V^2+\varphi\Big)\\ &=m\mathbf{v}\dot{\mathbf{v}}+M\mathbf{V}\dot{\mathbf{V}}+(\mathbf{v}-\mathbf{V})\nabla\varphi\\ &=(\mathbf{v}-\mathbf{V})(m\dot{\mathbf{v}}+\nabla\varphi)\\ &=-\frac{2}{3}\frac{e^2}{c^3}\dot{\mathbf{v}}^2.\end{aligned} \tag{6}$$

Now the total force $\mathbf{f}_{tot}$ is the sum of the force $\mathbf{f}_{M\to m}$ which the source particle M exerts on particle m, and the radiation reaction force $\mathbf{f}_{rad}$, i.e. $\mathbf{f}_{tot} = \mathbf{f}_{M\to m} + \mathbf{f}_{rad}$. Using the equation of motion $m\dot{\mathbf{v}} = \mathbf{f}_{tot}$ and the relation $\nabla\varphi = -\mathbf{f}_{M\to m}$, one gets $m\dot{\mathbf{v}} + \nabla\varphi = \mathbf{f}_{tot} - \mathbf{f}_{M\to m} = \mathbf{f}_{rad}$, and we are left with

$$(\mathbf{v} - \mathbf{V})\mathbf{f}_{rad} = -\frac{2}{3}\frac{e^2}{c^3}\dot{\mathbf{v}}^2. \tag{7}$$

The auxiliary potential φ has completely disappeared. Thus the equation for the radiation force becomes

$$\mathbf{f}_{rad} = -\frac{2}{3}\frac{e^2}{c^3}\dot{\mathbf{v}}^2\frac{\mathbf{v} - \mathbf{V}}{(\mathbf{v} - \mathbf{V})^2}. \tag{8}$$

One could have added a term $\mathbf{f}_\perp$ which is orthogonal to $\mathbf{v} - \mathbf{V}$. But from the vectors which are available one can only construct $(\mathbf{v} - \mathbf{V}) \times (\dot{\mathbf{v}} - \dot{\mathbf{V}})$ which is an axial vectors and therefore forbidden by parity.

Clearly invariance under non relativistic boosts is now restored, and the reason for the failure in the usual derivation is also clear. Performing the same steps as before, but eliminating the charged particle m instead of the source M, one finds that there is not only a force on the particle under consideration but also a force on the source particle. Using the unphysical concept of an external field ignores one of the fundamental laws of physics: Isaac Newton's third law *actio = reactio.*

The message learned from the above considerations is: **It is not possible to determine the radiation reaction force without knowing the motion of the source which causes the acceleration.**

But now a curiosity arises, in our opinion not a paradox, but nevertheless a rather strange consequence. Consider, for instance, a linear potential, i.e. a constant force. This can be approximately created by a far away source M. Imagine we could measure the radiation reaction force in a lab. Then, by comparison with (8), we could determine the relative velocity between particle and source, even without seeing the source! The situation becomes even more confusing if the force is due to several sources which are in motion with respect to each other. Plenty of conceptual problems to think about.

We will first give two simple applications for one dimensional problems and compare our results with those from the non relativistic LAD equation (4). For applications we take, of course, the limit $M/m \to \infty$ and can accordingly choose a convenient system in which $X = const.$ and $V = 0$, such that φ becomes a function of x only.

The equation of motion for one dimensional problems then reads

$$m\ddot{x} = -\varphi'(x) - m\tau_c\frac{\ddot{x}^2}{\dot{x}}, \tag{9}$$

where τ_c denotes the characteristic time

$$\tau_c = \frac{2}{3}\frac{e^2}{c^3 m}, \tag{10}$$

not to be confused with the proper time τ in the following section. Solving the quadratic equation (9) for $\ddot{x}$ gives

$$\ddot{x} = -\frac{\dot{x}}{2\tau_c}\left(1 \pm \sqrt{1 - \frac{4\tau_c\varphi'}{m\dot{x}}}\right) \to \begin{Bmatrix} -\dot{x}/\tau_c \\ -\varphi'/m \end{Bmatrix} \text{ for } \tau_c \to 0. \tag{11}$$

Only the lower sign for the square root gives the correct limit for $\tau_c \to 0$, therefore we will always choose this sign in the following. One also realizes the possibility of a complex square root. Note that the differential equation is non linear, we don't have the possibility of constructing real solutions by superposition. We will clarify the origin of this phenomenon when treating the example of a constant external force.

2.1 Constant external force

We consider a charge in a field with constant acceleration $-g$. The equation of motion is (this equation was also used by Hammond [5], without, however, mentioning the classically forbidden regions)

$$m\dot{v} = -mg - m\tau_c\frac{\dot{v}^2}{v}, \tag{12}$$

or

$$\dot{v} = -\frac{v}{2\tau_c}(1 - \sqrt{1 - 4g\tau_c/v}). \tag{13}$$

This is a first order separable differential equation for $v(t)$. It can be explicitly integrated to obtain t as function of v, but the result is not very enlightening. A qualitative discussion is more informative.

For negative v the differential equation is perfectly well behaved. This corresponds to the case that we simply drop the charge in the (say) gravitational field. For positive v, however, i.e. if we throw the charge upwards, there is no longer a real solution if $0 < v < v_c$, where we introduced the critical velocity

$$v_c = 4g\tau_c. \tag{14}$$

This behavior, though very strange at first sight, can be well understood. If the charge moves upwards it gains potential energy, furthermore it has to provide the energy for the radiation. These two energies have to be compensated by a corresponding loss of kinetic energy. But near the turning point the velocity is so small that it is no longer possible to decrease the kinetic energy sufficiently. There is no longer a solution for the energy requirement, one would need a negative kinetic energy. The mathematics consequently answers with a complex solution! For negative velocities there is no problem. When the particle falls downwards it looses potential energy which can be used to provide the energy for the radiation.

Clearly classical physics breaks down here. We are confronted with a typical tunneling problem, where the particle has to tunnel through a classically forbidden region. We also expect a certain probability for reflection instead of transmission. For all practical

situations one may safely ignore this (extremely small!) forbidden region. We did not attempt a quantum mechanical calculation of the tunneling. The problem is awkward, due to the lack of a Lagrangian or Hamiltonian formulation.

The problematic of the classically forbidden region disappears in a perturbative treatment of first order in τ_c. More precisely, because (12) can be written in the parameter free form $d\tilde{v}/d\tilde{t} = -1 - (d\tilde{v}/d\tilde{t})^2/\tilde{v}$, with $\tilde{t} = t/\tau_c$, $\tilde{v} = v/g\tau_c$, this means that $t \gg \tau_c$ with t chosen such that the turning occurs near $t = 0$. One finds the perturbative solution

$$\dot{v}(t) = -g(1 - \tau_c/t). \tag{15}$$

This implies that $x(t)$ is finite everywhere, $v(t)$ has a logarithmic singularity, and $\dot{v}(t)$ a pole. If one keeps away from the classically forbidden region one has an excellent approximation to the exact solution.

Of course one can also check the energy balance $\Delta E_{pot} + \Delta E_{kin} + \Delta E_{rad} = 0$ in first order of τ_c explicitly. The frequently asked question "Where does the energy come from if the particle permanently radiates?" has a simple answer. The radiation has the effect that, at a given position, the particle has a smaller velocity than it would have without radiation, or, formulated the other way round, it reaches a certain velocity only at a lower altitude.

We compare with the LAD equation (4), $m\dot{v} = -mg + m\tau_c\ddot{v}$, or $a = -g + \tau_c\dot{a}$. It has the solution $a = -g + a_i \exp(t/\tau_c)$, i.e. either a non radiating solution for $a_i = 0$, or a run away solution for $a_i \neq 0$.

2.2 Harmonic oscillator

The equation of motion is

$$m\ddot{x} = -m\omega^2 x - m\tau_c\ddot{x}^2/\dot{x}. \tag{16}$$

An ansatz of the form $x(t) = x_i \exp(\lambda t)$ leads to two complex solutions but these are useless for constructing a real solution because the equation is non linear and we are not allowed to build a superposition.

Remembering the previous discussion we expect that there will be no real solution close to the extrema where the velocity is small. Again the problem is less drastic in lowest order perturbation theory. One finds

$$x(t) = x_0 \cos\omega t \exp[-\tau_c(\omega^2 t/2 - \omega \tan\omega t \ln|\sin\omega t|)]. \tag{17}$$

We compare with the LAD equation $m\ddot{x} = -m\omega^2 x + m\tau_c\dddot{x}$. This is easily solved by the ansatz $x(t) = x_0 \exp(\lambda t)$ with λ a solution of the cubic equation $\lambda^2 + \omega^2 - \tau_c\lambda^3 = 0$. There is always one positive solution for λ which represents the run away solution. Furthermore there is a pair of complex conjugate solutions with negative real part which lead to damped oscillations. If τ_c is treated in lowest order one finds $\lambda = \pm i\omega - \tau_c\omega^2/2$. This term is also present in our perturbative solution (17), but the latter contains further terms which take into account the varying radiation during the oscillation. The LAD equation, on the other hand, averages the whole process.

If we ignore the subtleties near the classically forbidden regions around the extrema we can derive a nice general property. Consider two consecutive zeros, located at t_1 and $t_2 = t_1 + T/2$. While $x_1 = x_2 = 0$, the velocities are different, $v_2 = -\exp(-\alpha/2)v_1$, say. But since (16) is a second order differential equation the solution is fixed by the initial values x_i, v_i. Furthermore, if $x(t)$ is a solution of (16), then this is also the case for $const \cdot x(t)$. Therefore one can immediately conclude $x(t + T/2) = -\exp(\alpha/2)\, x(t)$, or, consequently,

$$x(t + nT/2) = (-1)^n \exp(-n\alpha/2)\, x(t) \text{ for any integer } n. \tag{18}$$

Knowledge within one half period is sufficient to know the whole solution.

2.3 Circular motion

Consider a charged particle which is forced to move in a circle of fixed radius r, and no other external forces present. For the radial and azimuthal components of velocity and acceleration one has, using the simplification $\dot{r} = 0$,

$$v_r = 0, \quad a_r = -{v_\varphi}^2/r, \quad a_\varphi = \dot{v}_\varphi. \tag{19}$$

Therefore $a^2 = v_\varphi^4/r^2 + {\dot{v}_\varphi}^2$. This leads to the following differential equation for $v_\varphi \equiv v$:

$$m\dot{v} = -m\tau_c(\frac{v^3}{r^2} + \frac{\dot{v}^2}{v}), \tag{20}$$

or

$$\dot{v} = -\frac{v}{2\tau_c}(1 - \sqrt{1 - 4{\tau_c}^2 v^2/r^2}). \tag{21}$$

The radicand is positive for any non relativistic motion and macroscopic radius. We don't look for exact solutions but are content with an expansion in τ_c to first order where we have to solve $\dot{v} = -\tau_c v^3/r^2$, resulting in

$$v = v_i/\sqrt{1 + 2\tau_c {v_i}^2 (t - t_i)/r^2}. \tag{22}$$

In lowest order of τ_c and for $v \ll c$ this agrees with the result of Hammond [1] as well as with the solution from the LAD equation. This is not surprising because in this example approximately $\dot{\mathbf{v}} \perp \mathbf{v}$, therefore the neglection of the integrated terms $\dot{\mathbf{v}}\mathbf{v}|_{t_1}^{t_2}$ is justified.

3. Relativistic equation

The relativistic generalization of the Larmor formula (1) reads

$$\frac{dP^\mu_{rad}}{d\tau} = -Ku^\mu, \tag{23}$$

with

$$K = -\frac{2}{3}\frac{e^2}{c^3}\frac{du_\nu}{d\tau}\frac{du^\nu}{d\tau} = m\tau_c\,[(\frac{(d\mathbf{v}/d\tau)^2}{1-v^2} + \frac{(\mathbf{v}d\mathbf{v}/d\tau)^2}{(1-v^2)^2}], \tag{24}$$

where $d\tau = \sqrt{1-v^2}\,dt$ denotes the proper time of the particle, and $u^\mu = (1/\sqrt{1-v^2})(1, \mathbf{v})$ the four-velocity. With the exception of the coefficient in K we put $c = 1$ everywhere.

We closely follow the non relativistic treatment. For deriving the formula we use a simple model with an instantaneous potential $\varphi(\mathbf{x} - \mathbf{X})$. This model does not claim any physical relevance. It is translation invariant but not Lorentz invariant. Nevertheless it will lead us to a covariant formula for the reaction force. The potential $\varphi(\mathbf{x} - \mathbf{X})$ is only an auxiliary construct. As in the non relativistic case it will completely disappear in the final formula.

The momentum and energy of our system are

$$\mathbf{P} = \frac{m}{\sqrt{1-v^2}}\mathbf{v} + \frac{M}{\sqrt{1-V^2}}\mathbf{V}, \tag{25}$$

$$P^0 = \frac{m}{\sqrt{1-v^2}} + \frac{M}{\sqrt{1-V^2}} + \varphi(\mathbf{x} - \mathbf{X}). \tag{26}$$

Now apply the time derivative $d/dt = \sqrt{1-v^2}\,d/d\tau$ to these equations. For the derivation we can specialize to a one dimensional motion where the formulae simplify. The differentiation can be easily performed, the result has to be identical to $-K\sqrt{1-v^2}\,u^\mu = -K(1, v)$. This leads to the two equations

$$\frac{m}{(1-v^2)^{3/2}}\frac{dv}{dt} + \frac{M}{(1-V^2)^{3/2}}\frac{dV}{dt} = -Kv, \tag{27}$$

$$\frac{m}{(1-v^2)^{3/2}}v\frac{dv}{dt} + \frac{M}{(1-V^2)^{3/2}}V\frac{dV}{dt} + (v-V)\varphi' = -K. \tag{28}$$

Eliminating the term with M from the momentum equation (27) and introducing into the energy equation (28) one obtains

$$(v-V)\Big(\frac{m}{(1-v^2)^{3/2}}\frac{dv}{dt} + \varphi'\Big) = -K(1-Vv). \tag{29}$$

We now work with the four-forces. As in the non relativistic case we have $F_{tot} = F_{M\to m} + F_{rad}$, and we can use the equation of motion for F_{tot} and the connection between φ' and $F_{M\to m}$,

$$F_{tot} = \frac{m}{(1-v^2)^2}\frac{dv}{dt}, \text{ and } \varphi' = -\sqrt{1-v^2}F_{M\to m}, \tag{30}$$

therefore

$$\Big(\frac{m}{(1-v^2)^{3/2}}\frac{dv}{dt} + \varphi'\Big) = \sqrt{1-v^2}(F_{tot} - F_{M\to m}) = \sqrt{1-v^2}F_{rad}. \tag{31}$$

One is left with

$$F_{rad} = -K\frac{1 - Vv}{\sqrt{1 - v^2}(v - V)}. \tag{32}$$

This can be written in a manifestly covariant form. In the non relativistic limit the force is proportional to the velocity, therefore F^{μ}_{rad} has to contain a term proportional to the four-velocity u^{μ}. In order to fulfill the condition $u_{\mu}F^{\mu}_{rad} = 0$ it must appear in the combination $u^{\mu} - U^{\mu}/Uu$. One has

$$u^m - U^m/Uu = \frac{v^m - V^m + v^2V^m - \mathbf{V}\mathbf{v}v^m}{\sqrt{1 - v^2}(1 - \mathbf{V}\mathbf{v})} \tag{33}$$

and

$$(u^{\mu} - U^{\mu}/Uu)^2 = -\frac{(\mathbf{v} - \mathbf{V})^2 - V^2v^2 + (\mathbf{V}\mathbf{v})^2}{(1 - \mathbf{V}\mathbf{v})^2}. \tag{34}$$

Therefore (32), which refers to the special case $\mathbf{V} \parallel \mathbf{v}$, can be written as the spatial part of

$$F^{\mu}_{rad} = \frac{K}{(u^{\nu} - U^{\nu}/uU)^2}(u^{\mu} - U^{\mu}/Uu), \tag{35}$$

with K given in (24). It is important to emphasize that the formulation of F^{μ}_{rad} as a four-force is only possible if one has the four-vector U^{μ} available. We are confident that this solution is unique.

For applications we will of course again consider the limit $M/m \to \infty$, so we can put $U^{\mu} \approx (1, 0)$. The spatial component of the reaction force (35) then simply becomes

$$F^m_{rad} = K\frac{u^m}{(u^{\nu} - U^{\nu}/Uu)^2} = -\frac{K}{\sqrt{1 - v^2}}\frac{v^m}{v^2}. \tag{36}$$

We give a simple application.

4. Constant external electric field

We consider a linear motion in a constant electric field $-E$ with $E > 0$. The minus sign was chosen for convenient comparison with the non relativistic case discussed previously. For $\mathbf{v} \parallel \dot{\mathbf{v}}$ the formula (24) for K simplifies to $K = m\tau_c(dv/d\tau)^2/(1 - v^2)^2$, and the equation of motion, conveniently written in terms of t, not of τ, reads

$$m\frac{dv}{dt} = -(1 - v^2)^{3/2}eE - m\tau_c\frac{(dv/dt)^2}{(1 - v^2)^{3/2}\,v}. \tag{37}$$

This corresponds to the previous non relativistic equation (12) with the replacements $g \to (1 - v^2)^{3/2}eE/m$, $\tau_c \to \tau_c(1 - v^2)^{-3/2}$. We can again solve for dv/dt and, performing these replacements in (14), obtain the critical velocity $v_c = 4eE\tau_c/m$, and the classically forbidden region $0 < v < v_c$.

The solution of (37) without the radiation term is well known. In first order perturbation theory in τ_c the solution of the full equation is

$$v(t) = -\Big[\frac{eEt}{q} - \tau_c meE(\frac{1}{q^2} + \frac{m}{2q^3}\ln\frac{q-m}{q+m})\Big], \tag{38}$$

with

$$q = \sqrt{m^2 + (eE)^2 t^2}. \tag{39}$$

The limit for large times is particularly simple:

$$v(t) \to -[1 - \frac{m^2}{2(eE)^2}(1 + \frac{2\tau_c eE}{m})\frac{1}{t^2}]. \tag{40}$$

It shows how the approach of v to -1 is slowed down by the radiation.

Summary and conclusions

We summarize our essential results and the properties of the equations derived here.

- It is mandatory to take into account the presence of the source which causes the acceleration of the charged particle. Only then one ends up with a force with the correct transformation property.
- There can be (extremely small) regions where the velocity is so small that the energy requirements cannot be fulfilled. The charge has to pass through such regions by quantum mechanical tunneling. It is impressive how the equation responds to the appearance of classically forbidden regions.
- Because we chose a direct approach without manipulations like partial integrations our equation does not contain third derivatives $\dddot{x}$, in clear contrast to the LAD equation. Consequently there are no problems with run away solutions.
- The non relativistic equation for the radiation reaction force can be considered as unique, because it was derived in a direct straight forward way. We believe that this is also the case for the relativistic generalization.
- In all examples considered so far we obtained reasonable results.

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General Relativity Extended to non-Riemannian Space-time Geometry

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Abstract: The gravitation equations of the general relativity, written for Riemannian space-time geometry, are extended to the case of arbitrary (non-Riemannian) space-time geometry. The obtained equations are written in terms of the world function in the coordinateless form. These equations determine directly the world function, (but not only the metric tensor). As a result the space-time geometry appears to be non-Rieamannian. Invariant form of the obtained equations admits one to exclude influence of the coordinate system on solutions of dynamic equations. Anybody, who trusts in the general relativity, is to accept the extended general relativity, because *the extended theory does not use any new hypotheses. It corrects only inconsequences and restrictions of the conventional conception of general relativity.* The extended general relativity predicts an induced antigravitation, which eliminates existence of black holes.

Keywords: General Relativity; Non-Riemannian Space-time Geometry; Gravitation Equations; World Function
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1. Introduction

In this paper we consider dynamic equations for the gravitational field which are obtained at the generalization of the relativity theory on the case of the most general space-time geometry. The general relativity supposes, that the Riemannian geometry is the most general space-time geometry. This supposition is based on our insufficient knowledge of a geometry, when one supposes that any geometry is axiomatizable. It means, that any

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geometry is constructed as a logical construction. In reality there exist nonaxiomatizable space-time geometries [1], which are constructed by means of the deformation principle [2] as a deformation of the proper Euclidean geometry. This geometry is described completely by the world function [3] and only by the world function. Such a geometry is called a physical geometry, because physicists need such a geometry, which is a science on location of geometrical objects and on their shape, (but not as a logical construction). Physicists use a geometry as a tool for investigation of the space-time properties. Physicists are indifferent to the question, whether or not a geometry is a logical construction.

Physical geometry may be continuous, or discrete. It may be even granular, i.e. partly continuous and partly discrete. The physical geometry is described by the same manner in all cases. Properties of the physical geometry are determined only by properties of the world function (but not by properties of the point set, where the geometry is given). As a result the physical geometry may be formulated in the coordinateless form (only in terms of the world function). A good illustration of this fact is the following example.

Let the proper Euclidean geometry be given in the Cartesian coordinates (x, y) on the square $[0,1] \times [0,1]$. It means, that the world function is given on this square. Let us map the square $[0,1] \times [0,1]$ onto the one-dimensional segment $[0,1]$ described by the coordinate X. Let the mapping $(x, y) \rightarrow X$ be one-to-one. It is possible only, if the mapping is discontinuous at any point. For instance, the mapping can be realized as follows. Let coordinates x, y, X be presented in the form of decimal fractions

$$x = 0.\alpha_1\alpha_2\alpha_3...., \qquad y = 0.\beta_1\beta_2\beta_3...., \qquad X = 0.\alpha_1\beta_1\alpha_2\beta_2\alpha_3\beta_3... \tag{1}$$

where α and β are decimal ciphers. The formulas (1) realizes one-to-one mapping $(x, y) \leftrightarrow X$. Now the world function σ is given on one-dimensional segment $[0,1]$.

$$\sigma(X_1; X_2) = \sigma(x_1, y_1; x_2, y_2)$$

Nevertheless, considering the world function on the one-dimensional segment $[0,1]$, one can reconstruct the proper Euclidean geometry. In particular, one can determine, that the geometry on the segment $[0,1]$ is the two-dimensional Euclidean geometry (in the sense, that the maximal number of linear independent vectors is equal two), although the geometry is given on one-dimensional segment.

For construction of a physical geometry it is sufficient to give a world function for any pair of points of the point set, where the geometry is given. One does not need to prove numerous theorems and to test a compatibility of geometric axioms. The world function $\sigma(P,Q) = \frac{1}{2}\rho^2(P,Q)$ is a function of two points P and Q, where $\rho(P,Q)$ is a distance between the two points. The number of possible world functions is much more, than the number of infinitesimal intervals $dS^2 = g_{ik}(x)\,dx^i dx^k$, which are functions of only one point. In particular, there is only one isotropic uniform geometry (the geometry of Minkowski) in the set of Riemannian geometries. It is described by the world function σ_{M}. In the set of physical geometries any geometry is isotropic and uniform, if it is described by the world function $\sigma = F(\sigma_{\mathrm{M}})$, where F is an arbitrary function, having the property $F(0) = 0$ and σ_{M} is the world function of the geometry of Minkowski.

In particular, the space-time geometry, described by the world function

$$\sigma = \sigma(\sigma_{\mathrm{M}}) = \sigma_{\mathrm{M}} + \lambda_0^2 \mathrm{sgn}(\sigma_{\mathrm{M}}), \qquad \lambda_0^2 = \frac{\hbar}{2bc} = \mathrm{const} \tag{2}$$

is uniform and isotropic. Here $\hbar$ is the quantum constant, c is the speed of the light and b is some universal constant. Besides, this geometry is nonaxiomatizable and discrete. In this space-time geometry any motion of a free pointlike particles is multivariant (stochastic). Statistical description of this stochastic motion is equivalent to the quantum description in terms of the Schrödinger equation [4]. This circumstance admits one to obtain a statistical foundation of quantum mechanics and to interpret quantum effects as geometrical effects. It admits one to exclude the quantum principles from the set of prime physical principles and to reduce the number of physical essences, what is important for fundamental physical theories.

By definition the special relativity is a consideration of physical phenomena in the flat uniform isotropic space-time. In the set of Riemannian geometries there is only one such a geometry. It is the space-time geometry of Minkowski. Description of physical phenomena in the geometry (2) should be qualified as an extended special relativity, because the space-time geometry (2) is isotropic and uniform, but it is non-Riemannian.

Statistical foundation of quantum theory shows also, that the real space-time geometry may be non-Riemannian, and one cannot restrict oneself, considering only the Riemannian space-time geometries.

Generalization of the general relativity on the case of physical space-time geometry appears to be possible only at taking into account two essential clauses:

(1) Consideration of physical geometries.
(2) Use of adequate relativistic concepts, and, in particular, a use of the relativistic concept of the events nearness.

The reasons of violation of the first condition are investigated in [5].

In the beginning of the twentieth century the theoretical physics developed on the way of geometrization. The special relativity and the general relativity were only stages of this geometrization. But the physics geometrization appeared to be restricted by our poor knowledge of geometry, when one knew only axiomatizable geometries. One was not able to work with discrete geometries and geometries with restricted divisibility. Quantum effects might be explained easily by multivariance of the space-time geometry. However, the property of multivariance [6] was not known in the beginning of the twentieth century, and scientists were forced to introduce new (quantum) principles of dynamics. As a result the quantum paradigm of the microcosm physics development appeared. The quantum paradigm dominated during the whole twentieth century. The quantum paradigm contains more essences, than it is necessary.

In the end of the twentieth century, when our knowledge of geometry became more complete, we may return to the program of further geometrization of physics. The geometrical paradigm appeared to be possible, when, using classical dynamic principles, quantum effects are freely explained by the properties of the space-time geometry. The geometrical paradigm is more attractive, because it uses less essences, than the quantum

paradigm does. To replace the quantum paradigm by the geometrical one, it is necessary to generalize the special relativity and the general relativity on the case of an arbitrary physical geometry of space-time.

The physical geometry is a geometry, which is described completely by the world function, (world function is a half of the squared distance). Practically, the physical geometry is a metric geometry, which is liberated from all constraints on metric except the constraint, that the world function (or metric) is equal to zero for two coinciding points. The physical geometry is a very simple construction [1, 2]. For constructing the physical geometry one does not need to formulate axioms and to prove numerous theorems. It is sufficient to know the proper Euclidean geometry, which is used as a standard physical geometry. All definitions of the proper Euclidean geometry $\mathcal{G}_{\mathrm{E}}$ may be formulated in terms of the Euclidean world function σ_{E}. Replacing the Euclidean world function σ_{E} in all definitions of the Euclidean geometry by the world function σ of the physical geometry $\mathcal{G}$, one obtains all definitions of the physical geometry $\mathcal{G}$, described by the world function σ.

Besides, the physical geometry is a monistic conception, which is described by the only fundamental quantity σ. All other geometrical quantities and concepts are expressed via fundamental quantity automatically. This circumstance admits one to modify a physical geometry easily, because all other geometric quantities concepts are modified automatically at modification of the fundamental quantity σ. [7]. Program of physics geometrization admits one to construct a monistic conception of physics with the fundamental quantity σ.

The set of all Riemannian geometries is only a small part of the set of all physical geometries. A generalization of the relativity theory on the case of arbitrary physical geometry admits one to obtain such results, which cannot be obtained in the framework of the Riemannian geometry. The generalization of the special relativity (motion of particle in the given space-time geometry) on the case of arbitrary physical space-time geometry has been made already [8]. A generalization of description of the matter influence on the arbitrary space-time geometry met some problems. These problems are connected with the fact, that in the relativity theory some basic concepts are taken from the nonrelativistic physics. Concepts of nonrelativistic physics are inadequate for consecutive geometric description of the relativity theory and for generalization of this description on the case of a more general space-time geometry.

Practically all physical phenomena on the Earth are nonrelativistic. At first, we study nonrelativistic physics with its nonrelativistic concepts. Relativistic effects appeared as corrections to nonrelativistic physics. In the beginning of the twentieth century the relativistic physics was presented in terms of slightly corrected nonrelativistic concepts. For instance, the relativity principle has been presented as invariance of the dynamic equations with respect to Lorentz transformation and as existence of the supreme speed of the interaction propagation. Such a formulation is useful for pedagogical goals, when one needs to transit from concepts of nonrelativistic physics to relativistic ones. However, such a formulation is not effective, when one tries to develop the relativistic physics.

In this case one should use concepts, which are adequate for relativistic physics. In particular, the relativity principle is formulated in adequate concepts as follows. The relativistic physics is a physics in the pseudo-Euclidean space-time geometry of index 1 (geometry of Minkowski or that of Kaluza-Klein). All other details of description are corollaries of properties of the space-time geometry. For instance, properties, concerning the role of the light speed, are pure geometrical properties of the space-time.

Unfortunately, the formulation of the relativity theory in adequate (geometrical) terms is used rare. The main difference of space-time geometry in a relativistic theory from that in the non-relativistic (Newtonian) physics is as follows. Relativistical event space (space-time) geometry is described by one invariant (space-time interval), whereas in the Newtonian physics the event space is described by two invariants (spatial distance and temporal interval). Sometimes one does not mention this difference in textbooks. Instead one speaks on difference in transformation laws. In reality, the difference in the number of invariants is a fundamental property, whereas the difference of the transformation laws is a very special property, because it is essential only for flat space-time geometry. Besides, the transformational properties are used only at description at some coordinate system. They are useless at the coordinateless description. This difference of formulations is not essential, when the theory is used for calculation of concrete physical effects. However, this difference becomes essential, if one tries to obtain a generalization of the relativity theory on the case of the arbitrary space-time geometry.

For instance, the concept of a pointlike particle as a point in the configuration space is a nonrelativistic concept. It needs concepts of velocity and acceleration of this particle. These concepts are secondary concepts, which can be introduced only after introduction of the linear vector space and, in particular of a coordinate system. These concepts are inadequate in the case of a discrete space-time geometry. As a result the concept of velocity and that of acceleration cannot be used in an extension of the relativity theory to the case of arbitrary space-time geometry, which may be discrete.

In the general relativity all interactions (electromagnetic and gravitational) are supposed to be short-range interactions. Concept of short-range interaction is based on the nonrelativistic concept of the events nearness. The events are considered as points in the event space (space-time). Two events are considered to be near, if they happen in the same place at the same time moment. This definition of nearness of events is nonrelativistic, because this definition refers to a spatial distance and to a temporal interval at once. A consistent relativistic concept of nearness is to contain a reference to only quantity: space-time interval, (or world function). For instance, if a supernew star flashed very far, and an observer on the Earth observed this flash, the event of flash and the event of this flash observation on the Earth are near (close) events.

According to common viewpoint the statement on nearness of the two events (flash and observation of this flash) seems to be rather strange and unexpected. However, from consistent relativistic viewpoint the two events are near, because space-time interval between them is equal to zero.

The problem of relativistic concept of nearness is discussed in [9]. It is known as the

principle of Fokker [10], which is interpreted as a conception of the action at a distance (but not as a relativistic concept of nearness). The action at a distance is treated as a direct influence of one object onto another one without intermediate agent circulatory between them.

2. Relativistic Concept of Nearness

Let us consider the proper Euclidean geometry. Let $\rho(P,Q)$ be the Euclidean distance between the points P and Q. The set O_ε of points P, defined by the relation

$$O_\varepsilon = \{P|\ \rho(O,P) < \varepsilon\}, \qquad \varepsilon > 0 \tag{1}$$

is called ε-vicinity of the point O. If the parameter ε is small, the points P and O are near $(P \simeq O)$. If $\varepsilon \to 0$, ε-vicinity O_ε degenerates into one point $O_0 = O$. It is easy to see, that, if $P \simeq Q$, then $Q \simeq P$.

The relation of nearness in the proper Euclidean geometry has the property of transitivity: If $P \in O_\varepsilon$ and $Q \in O_\varepsilon$, then $P \in Q_{2\varepsilon}$ and $Q \in P_{2\varepsilon}$. It follows from the triangle axiom,

$$\rho(O,P) + \rho(O,Q) \geq \rho(Q,P)$$

which is valid for the proper Euclidean geometry. If $\varepsilon \to 0$, then $2\varepsilon \to 0$ also. It means that, if $P \simeq O$ and $Q \simeq O$, then $P \simeq Q$.

The property of transitivity seems to be a natural property of the relation of nearness. However, the transitivity property of the nearness relation does not take place in the space-time geometry, for instance, in the geometry of Minkowski. In this case the ε-vicinity O_ε of the point O is defined by the relation

$$O_\varepsilon = \{R|\ |\rho(O,R)| < \varepsilon\}, \qquad \rho(O,R) = \sqrt{2\sigma_{\mathrm{M}}(O,R)} \tag{2}$$

Here $\sigma_{\mathrm{M}}(P,Q) = \sigma_{\mathrm{M}}(x,x')$ is the world function of the space-time of Minkowski

$$\sigma_{\mathrm{M}}(P,Q) = \sigma_{\mathrm{M}}(x,x') = \frac{1}{2}(g_{\mathrm{M}})_{ik}\left(x^i - x'^i\right)\left(x^k - x'^k\right) \tag{3}$$

x, x' are coordinates of points P and Q in some inertial coordinate system, and $(g_{\mathrm{M}})_{ik}$ is the metric tensor in this coordinate system.

In this case the points with coordinates $P = \left\{\sqrt{a^2+\varepsilon^2}, a, 0, 0\right\}$ and $Q = \left\{\sqrt{a^2+\varepsilon^2}, -a, 0, 0\right\}$ belong to ε-vicinity of the point $O = \{0,0,0,0\}$, whereas $P \notin Q_{2\varepsilon}$, because

$$|2\sigma(P,Q)| = \left|\rho^2(P,Q)\right| = 4a^2 \tag{4}$$

As far as the quantity a may be indefinitely large, the spatial distance between the points P and Q may be very large, although both points are near to the point O ($P \simeq O$ and $Q \simeq O$).

In the space-time of Minkowski the ε-vicinity of the point $O = \{0,0,0,0\}$ is a region of the space-time between two hyperboloids

$$\left(x^0\right)^2 - \mathbf{x}^2 = \varepsilon^2, \qquad \left(x^0\right)^2 - \mathbf{x}^2 = -\varepsilon^2 \tag{5}$$

Formally the relation (5) determines a sphere of radius ε in the space-time of Minkowski. At $\varepsilon \to 0$ this region turns to the light cone with the vertex at the point O. Thus, at $\varepsilon \to 0$ in the proper Euclidean geometry the ε-vicinity of the point P is the same point P, whereas in the geometry of Minkowski the ε-vicinity of the point P is the light cone with the vertex at the point P.

In the nonrelativistic physics the ε-vicinity O_ε of the point $O = \{0, 0, 0, 0\}$ is defined by relations

$$O_\varepsilon = \left\{ \left\{ x^0, \mathbf{x} \right\} \mid \ \left| x^0 \right| < \varepsilon \wedge |\mathbf{x}| < \varepsilon \right\} \tag{6}$$

In the limit $\varepsilon \to 0$, the ε-vicinity (6) turns to one point O. Thus, in the non-relativistic physics there are only one near point, whereas in the relativistic physics there is a continual set of near points. This difference appears to be very important in definition of short-range interaction between particles.

Let us stress, that introducing cone-shaped ε-vicinity and nearness of points on the light cone to the vertex of the cone, **we do not suggest any hypothesis. We follow only the relativity principle.** If we follow the relativity principles, we should accept the fact of the cone-shaped ε-vicinity, because pointlike shape of the ε-vicinity in the limit $\varepsilon \to 0$ is a remnant of the nonrelativistic theory.

3. Relativistic Concept of a Pointlike Particle

In the consecutive geometric description any particle is realized by its skeleton. In the case of a pointlike particle the skeleton is the ordered set of two points $\{P_s, P_{s+1}\}$. The vector $\mathbf{P}_s\mathbf{P}_{s+1}$ describes the geometric momentum of particle. The length $|\mathbf{P}_s\mathbf{P}_{s+1}| = \sqrt{2\sigma\left(P_s, P_{s+1}\right)}$ of the vector $\mathbf{P}_s\mathbf{P}_{s+1}$ describes the geometric mass of particle. Such a description is a pure geometric one.

Motion of a pointlike particle is described by a world chain $\mathcal{C}$, consisting of connected links $\mathcal{T}_{[P_sP_{s+1}]}$

$$\mathcal{C} = \sum_s \mathcal{T}_{[P_sP_{s+1}]} \tag{1}$$

Any link $\mathcal{T}_{[P_sP_{s+1}]}$ is a segment of straight line, determined by the skeleton $\mathcal{P}_1^{(s)} = \{P_s, P_{s+1}\}$. The link $\mathcal{T}_{[P_sP_{s+1}]}$ is a set of points, determined by the relation

$$\mathcal{T}_{[P_sP_{s+1}]} = \left\{ R | \sqrt{2\sigma\left(P_s, R\right)} + \sqrt{2\sigma\left(R, P_{s+1}\right)} - \sqrt{2\sigma\left(P_s, P_{s+1}\right)} = 0 \right\} \tag{2}$$

The length

$$\mu = \sqrt{2\sigma\left(P_s, P_{s+1}\right)} \tag{3}$$

of all links is the same. The length μ is the geometric mass of the particle, which is connected with the usual mass m of the particle by the relation

$$m = b\mu = b\sqrt{2\sigma\left(P_s, P_{s+1}\right)} \tag{4}$$

where b is the same universal constant, which appears in (2)

Complicated (not pointlike) particles are described by a more complicated skeleton $\mathcal{P}_n = \{P_0, P_1, ...P_n\}$ [8]

Description of the particle motion does not need an introduction of a coordinate system. Details of such a description of the particle motion may be found in [8]. Such a description is generalized easily on the case of arbitrary space-time geometry (in particular, discrete one). In the microcosm the structure of the world chain (1) is essential, but outside the microcosm one may consider the length μ of a link $\mathcal{T}_{[P_s P_{s+1}]}$ to be infinitesimal, and to replace the world chain by a smooth world line.

Let $\mathcal{L}$ be a world line of a pointlike particle, and the point $P \in \mathcal{L}$. A set $\mathbb{N}_P$ of events Q, which are near to the point P is different from the relativistic viewpoint and from the nonrelativistic one. From nonrelativistic (conventional) viewpoint $\mathbb{N}_P = \{P\}$, whereas from the relativistic viewpoint $\mathbb{N}_P = \mathcal{C}_P$, where $\mathcal{C}_P$ is the light cone with the vertex at the point P.

$$\mathcal{C}_P = \{R | \sigma (P, R) = 0\} \tag{5}$$

It is known, that the electromagnetic interaction between two pointlike charged particles is carried out only via points, connecting with vanishing space-time interval (retarding interaction), i.e. via points, which are near from the relativistic viewpoint. The same is valid for the gravitational interaction. On the other hand, the near points of the world line $\mathcal{L}$ should be interpreted in a sense of points belonging to the world line. In this sense any interaction of two pointlike particles via near points may be interpreted as a direct interaction (collision).

The light cones with vertexes at the points $P \in \mathcal{L}$, may be considered as attributes of the pointlike particle, which is described by the world line $\mathcal{L}$. We shall consider these light cones, directed into the past, as bunches of isotropic straight lines $\mathcal{H}$. In other words, any world line $\mathcal{L}$ of a pointlike particle is equipped by bunches $\mathcal{C}_P$ of hair $\mathcal{H}_P$ at any point $P \in \mathcal{L}$. Any hair $\mathcal{H}_P$ consists of points $R \in \mathcal{H}_P$, which are near to the point $P \in \mathcal{L}$, $\quad (\sigma (P, R) = 0, \quad R \in \mathcal{H}_P)$ on the world line $\mathcal{L}$. The point P is a footing of the hair $\mathcal{H}_P$. The length of the hair $\mathcal{H}_P$ is equal to zero, because the hair $\mathcal{H}_P$ consists of points, which are near to the point P. Although the length of any hair is equal to zero, nevertheless the hairs of any world line cover the whole space-time. When some point $P' \in \mathcal{H}_P$, $P \in \mathcal{L}_1$ of the world line $\mathcal{L}_1$ hair coincides with a point $P' = P_2 \in \mathcal{L}_2$ of other world line $\mathcal{L}_2$, the particle $\mathcal{L}_2$ transfers a part of its momentum to the particle $\mathcal{L}_1$. See figure

What part of its momentum does the particle $\mathcal{L}_2$ transfer, depends on the point $P' \in \mathcal{H}_P$, which is a common point with $\mathcal{L}_2$ $(P' = P_2 \in \mathcal{L}_2)$.

Although the length of any part of the hair is equal to zero, nevertheless there is some invariant parameter along any hair $\mathcal{H}$. This parameter l_r is the relative length of the hair segment. The relative length (r-length) of the point P is the more, the "farther" the point $R \in \mathcal{H}_P$ lies from the footing P of the hair $\mathcal{H}_P$. The r-length l_r of the point $R \in \mathcal{H}_P$ is defined by the relation

$$l_r = l_r (P, R) = \frac{(\mathbf{PR}.\mathbf{Q}_0\mathbf{Q}_1)}{|\mathbf{Q}_0\mathbf{Q}_1|} \tag{6}$$

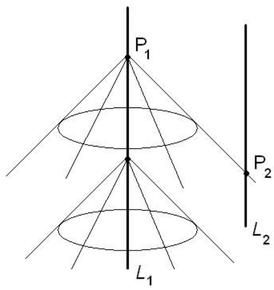

Fig. 1

where vector $\mathbf{Q}_0\mathbf{Q}_1$ is an arbitrary timelike vector ($\sigma(O_0, Q_1) > 0$). The scalar product $(\mathbf{PR}.\mathbf{Q}_0\mathbf{Q}_1)$ of vectors $\mathbf{PR}$ and $\mathbf{Q}_0\mathbf{Q}_1$ is defined by the relation

$$(\mathbf{PR}.\mathbf{Q}_0\mathbf{Q}_1) = \sigma(P, Q_1) + \sigma(R, Q_0) - \sigma(P, Q_0) - \sigma(R, Q_1) \tag{7}$$

$$|\mathbf{Q}_0\mathbf{Q}_1| = \sqrt{(\mathbf{Q}_0\mathbf{Q}_1.\mathbf{Q}_0\mathbf{Q}_1)} = \sqrt{2\sigma(Q_0, Q_1)} \tag{8}$$

It follows from expressions (6) - (8), that the relative length is invariant, because it is expressed in terms of the world function. Numerical value of the r-length depends on the choice of the timelike vector $\mathbf{Q}_0\mathbf{Q}_1$. Sign of the r-length depends on the choice of the timelike vector $\mathbf{Q}_0\mathbf{Q}_1$ also. However, the order of points on the hair, directed to the past, is determined single-valuedly by the value of the r-length.

If for some choice of the timelike vector $\mathbf{Q}_0\mathbf{Q}_1$

$$|l_r(P, R_1)| < |l_r(P, R_2)|, \qquad R_1, R_2 \in \mathcal{H}_P \tag{9}$$

then the relation (9) takes place for any other choice of timelike vector $\mathbf{Q}_0\mathbf{Q}_1$. It means that the point R_1 is located between the points P and R_2. The quantity of the transferred momentum is inversely to the r-length $l_r(P, P')$ between the footing of the hair $P \in \mathcal{L}_1$ and the point $P' \in \mathcal{L}_2$, $P' \in \mathcal{H}_P$.

The concept of the world line hair admits one to consider and to calculate electromagnetic and gravitational interaction of particles as a direct collision of one particle with a hair of other particle. As far as the hairs of a world line are considered as attributes of a particle, one may consider electromagnetic and gravitational interaction of particles as a direct collision of particles. Such a description of the particle interaction *does not mention about gravitational and electromagnetic fields. Such a description is a consecutive relativistic description.*

In the nonrelativistic theory the electromagnetic and gravitational fields are *essences, which exist independently of the matter.* These essences provide the momentum transfer from one particle to another one. Introduction of such essences is necessary, because the nonrelativistic concept of nearness is used. In the consecutive relativistic theory, which uses relativistic concept of nearness, one does not need to consider the electromagnetic and gravitational fields as additional essences. It is sufficient to consider them as a manner of description of particle interaction. The less number of essences is contained in a fundamental theory, the more perfect fundamental theory takes place.

Our conclusion, that gravitational and electromagnetic fields are not physical essences (they are only attributes of the world function) seems rather unexpected for most physicists. It is connected with the fact, that the relativity theory is considered usually as a correction to the nonrelativistic physics. As a result the relativity theory is presented almost in all textbooks in terms of concepts of nonrelativistic physics. The relativity theory is studied after presentation of nonrelativistic physics. It is natural, that the relativity theory is presented in terms of nonrelativistic concepts. Such a presentation is clearer for physicists, which know nonrelativistic physics. New specific relativistic concepts are used only in the case, when one cannot ignore them.

However, the relativity theory is a self-sufficient fundamental theory, which can and must be presented without a mention of nonrelativistic concepts. *Furthermore, the relativity theory can be developed successfully only in terms of adequate (relativistic) concepts.*

Let there be two timelike world lines $\mathcal{L}_1$ and $\mathcal{L}_2$ of two different particles. Any point $P \in \mathcal{L}_1$ corresponds, at least, to one near point $P' \in \mathcal{L}_2$, i.e. $P' \simeq P$, because the timelike world line $\mathcal{L}_2$ crosses the light cone with the vertex at the point $P \in \mathcal{L}_1$. In other words, any point of the world line $\mathcal{L}_1$ has a near point on the world line $\mathcal{L}_2$ and vice versa.

Let us consider the space-time Ω of Minkowski, which is described by the world function σ_{M}, defined by (3). Let the inertial coordinate system K be used, and the world chains $\mathcal{C}_1, \mathcal{C}_2$ be timelike. The world chains $\mathcal{C}_1$ and $\mathcal{C}_2$ consist of connected segments $\mathcal{T}_{[P_l P_{l+1}]}$ and $\mathcal{T}_{[P_l' P_{l+1}']}$

$$\mathcal{C}_1 = \bigcup_l \mathcal{T}_{[P_l P_{l+1}]}, \qquad \mathcal{C}_2 = \bigcup_l \mathcal{T}_{[P_l' P_{l+1}']} \tag{10}$$

$$\mathcal{T}_{[P_l P_{l+1}]} = \left\{ R | \sqrt{2\sigma_{\mathrm{M}} (P_l, R)} + \sqrt{2\sigma_{\mathrm{M}} (P_{l+1}, R)} = \sqrt{2\sigma_{\mathrm{M}} (P_l, P_{l+1})} \right\} \tag{11}$$

$$\mathcal{T}_{[P_l' P_{l+1}']} = \left\{ R | \sqrt{2\sigma_{\mathrm{M}} (P_l', R)} + \sqrt{2\sigma_{\mathrm{M}} \left(P_{l+1}', R\right)} = \sqrt{2\sigma_{\mathrm{M}} \left(P_l', P_{l+1}'\right)} \right\} \tag{12}$$

All segments of a world chain have the same geometrical length μ, defined by the relation (3) The real mass of the particle, described by the world chain, is connected with the geometric mass μ by means of the relation (4).

Outside the microcosm the length μ is small with respect to characteristic size of the world chain, and one may consider, that the vectors $\mathbf{P}_l\mathbf{P}_{l+1}$ of any link have infinitesimal length. In the Minkowski space-time Ω the timelike links $\mathcal{T}_{[P_l P_{l+1}]}$ are one-dimensional infinitesimal timelike segments. The timelike world chain $\mathcal{C}$ can be replaced by a smooth timelike world line $\mathcal{L}$, whose points are labelled by a parameter τ. The world line is described by the vector $\mathbf{PP}'(\tau)$, where P is the origin of the coordinate system K and $P'(\tau) \in \mathcal{L}$. The vectors $\mathbf{P}_l\mathbf{P}_{l+1}$ of links turn into infinitesimal vectors $\mathbf{P}'(\tau)\,\mathbf{P}'(\tau + d\tau)$, which are tangent to the world line.

Let the world lines $\mathcal{L}_1$ and $\mathcal{L}_2$ be timelike. For timelike world lines the infinitesimal segments $\mathcal{T}_{[P_l P_{l+1}]} = \mathbf{P}'(\tau)\,\mathbf{P}'(\tau + d\tau)$ are timelike, and the geometrical mass μ is real $(\sigma(P_l, P_{l+1}) > 0)$. In this case the world lines $\mathcal{L}_1$ and $\mathcal{L}_2$ are one-dimensional, and all points of a world line can be labelled by a parameter τ.

As far as the space-time Ω of Minkowski is a linear vector space, the vectors $\mathbf{PP}'(\tau)$ can be represented as a linear combination of basic vectors

$$\mathbf{PP}'(\tau) = f^k(\tau)\,\mathbf{e}_k, \tag{13}$$

where $\mathbf{e}_k$ are basic vectors of the inertial coordinate system K with the origin P. The functions $f^k(\tau)$, $k = 0,1,2,3$ are coordinates of points of the world line $\mathcal{L}_2$.

Four basic vectors $\mathbf{e}_k$ may be presented in the form

$$\mathbf{e}_k = \mathbf{PQ}_k, \qquad \mathbf{e}^i = (g_\mathrm{M})^{ik}\,\mathbf{e}_k = (g_\mathrm{M})^{ik}\,\mathbf{PQ}_k, \qquad k = 0,1,2,3 \tag{14}$$

Here and further a summation over repeating Latin indices is produced $0\div3$. The basic vector $\mathbf{e}_k = \mathbf{PQ}_k$ is determined by the origin point P and by the end point Q_k. Such a representation is necessary to use the scalar product in arbitrary physical geometry, where there is no linear space, and the scalar product of two vectors $\mathbf{PR}$ and $\mathbf{Q}_0\mathbf{Q}_1$ is defined by the relation (7). The scalar product $(\mathbf{PR}.\mathbf{Q}_0\mathbf{Q}_1)$ of two vectors $\mathbf{PR}$ and $\mathbf{Q}_0\mathbf{Q}_1$ is defined only via the world function without a reference to the properties of the linear vector space.

Coordinates of the points $P'(\tau)$ in the coordinate system K can be presented as follows

$$\mathcal{L}_2 : P'(\tau) = \left\{f^k(\tau)\right\} = \left\{\left(\mathbf{PP}'(\tau).\mathbf{e}^k\right)\right\} = \left\{(g_\mathrm{M})^{ik}\left(\mathbf{PP}'(\tau).\mathbf{e}_i\right)\right\}, \quad \tau\in\mathbb{R}, \quad P'\in\Omega \tag{15}$$

or

$$f^k(\tau) = (g_\mathrm{M})^{kl}\left(\mathbf{PP}'(\tau).\mathbf{e}_l\right) = (g_\mathrm{M})^{kl}\left(\mathbf{PP}'(\tau).\mathbf{PQ}_l\right) \tag{16}$$

$$f_k(\tau) = (g_\mathrm{M})_{kl}\,f^l(\tau) = \left(\mathbf{PP}'(\tau).\mathbf{PQ}_k\right) \tag{17}$$

where $(g_\mathrm{M})^{kl}$ is the contravariant metric tensor, which is obtained from the covariant metric tensor $(g_\mathrm{M})_{kl}$ by means of relations

$$(g_\mathrm{M})^{il}(g_\mathrm{M})_{lk} = \delta^i_k, \qquad (g_\mathrm{M})_{lk} = (\mathbf{e}_i.\mathbf{e}_k) = (\mathbf{PQ}_i.\mathbf{PQ}_k) \tag{18}$$

In reality the functions $f^k(\tau)$ are piecewise. But for simplicity we shall consider them as continuous and differentiable

$$\mathcal{L}_2 : \qquad x^k = f^k(\tau), \qquad \dot f^k(\tau) \equiv \frac{df^k(\tau)}{d\tau} \qquad k = 0,1,2,3, \tag{19}$$

4. Dynamic Equations for Calculation of World Function of Space-time

Variation δg_{ik} of the metric tensor, which is generated by particles in the space-time geometry of Minkowski is described by the relation [11]

$$\left(c^{-2}\partial_0^2 - \nabla^2\right)\delta g_{ik} = -\kappa T_{ik} \tag{1}$$

where T_{ik} is the energy-momentum tensor of particles. The constant $\kappa = 8\pi G/c^2$, where G is the gravitational constant and c is the speed of the light. Solution of this equation can be presented in the form

$$\delta g_{ik}(x) = -\kappa \int G_{\rm ret}(x, x') T_{ik}(x') \sqrt{-g_{\rm M}} d^4 x', \tag{2}$$

$$g_{\rm M} = \det ||(g_{\rm M})_{ik}||, \qquad i, k = 0, 1, 2, 3 \tag{3}$$

where the retarded Green function $G_{\rm ret}(x, x')$ has the form

$$G_{\rm ret}(x, x') = \frac{1}{2\pi c} \theta\left(x^0 - x^{0\prime}\right) \delta\left(2\sigma_{\rm M}(x, x')\right) \tag{4}$$

Here $\sigma_{\rm M}$ is the world function of the Minkowski space-time, defined by the relation (3), and the multiplier

$$\theta(x) = \begin{cases} 1 \text{ if } x > 0 \\ 0 \text{ if } x \leq 0 \end{cases} \tag{5}$$

Idea of derivation of dynamic equation for world function is very simple. It is based on the deformation principle [2]. Dynamic equations (1) for weak gravitational field in the space-time geometry of Minkowski are written in terms of the world function and only in terms of the world function. As far as these equations contain only world function, they are declared to be valid for variation of the world function of any physical space-time geometry under action of the matter added in the space-time.

The energy-momentum tensor T_{ik} of particles has the form

$$T^{ik}(x) = \sum_s p^i_{(s)}(x) u^k_{(s)}(x), \qquad i, k = 0, 1, 2, 3 \tag{6}$$

where $u^k_{(s)}(x)$ and $p^k_{(s)}(x)$ are distributions of the 4-velocity and of the 4-momentum of the sth particle in the space-time. We have for the particle number s

$$\mathcal{L}_{(s)}: \qquad x^i = f^i_{(s)}(\tau), \qquad p^i_{(s)} = b\dot{f}^i_{(s)}(\tau) d\tau, \qquad (g_{\rm M})^{ik} p_{(s)i} p_{(s)k} = m^2_{(s)} c^2, \tag{7}$$

$$u^k_{(s)} = \frac{\dot{f}^k(\tau)}{\sqrt{g_{rl} \dot{f}^r_{(s)}(\tau) \dot{f}^l_{(s)}(\tau)}} \tag{8}$$

$$p_{(s)i} = (g_{\rm M})_{ik} b \left(f^k_{(s)}(\tau + d\tau) - f^k_{(s)}(\tau)\right) = (g_{\rm M})_{ik} \dot{f}^k_{(s)}(\tau) b d\tau \tag{9}$$

where the constant b is the proportionality coefficient (4) between the length of the world line link $\mu = |\mathbf{P}_l \mathbf{P}_{l+1}|$ and the particle mass, described by this link. We have

$$p_{(s)i} g^{ik}_{\rm M} p_{(s)k} = g_{{\rm M}ik} \dot{f}^i_{(s)}(\tau) \dot{f}^k_{(s)}(\tau) b^2 (d\tau)^2 = m^2_{(s)} c^2 \tag{10}$$

$$m_{(s)} = \frac{b d\tau}{c} \sqrt{(g_{\rm M})_{rl} \dot{f}^r_{(s)}(\tau) \dot{f}^l_{(s)}(\tau)} \tag{11}$$

Then it follows from (7) and (11), that

$$p_{(s)}^{i}=\frac{m_{(s)}c\dot{f}^{i}}{\sqrt{(g_{\mathrm{M}})_{rl}\,\dot{f}_{(s)}^{r}\dot{f}_{(s)}^{l}}} \tag{12}$$

According (6) one obtains for the pointlike particles

$$T^{ik}(x)=\sum_{s}\frac{m_{(s)}c\dot{f}_{(s)}^{i}(\tau)\,\dot{f}_{(s)}^{k}(\tau)}{(g_{\mathrm{M}})_{rl}\,\dot{f}_{(s)}^{r}(\tau)\,\dot{f}_{(s)}^{l}(\tau)}\prod_{\alpha=1}^{\alpha=3}\delta_{\alpha}\left(x^{\alpha}-f_{(s)}^{\alpha}(\tau)\right) \tag{13}$$

where δ-function is defined by the relations

$$\int_{V}\prod_{\alpha=1}^{\alpha=3}F(\mathbf{x}')\,\delta_{\alpha}\left(x'^{\alpha}-f^{\alpha}(\tau)\right)\sqrt{-g_{\mathrm{sp}}}d\mathbf{x}'=\begin{cases}F(\mathbf{f}(\tau)) & \text{if } \mathbf{x}'\in V\\ 0 & \text{if } \mathbf{x}'\notin V\end{cases} \tag{14}$$

Here

$$g_{\mathrm{sp}}=\det\left|\left|(g_{\mathrm{M}})_{\alpha\beta}\right|\right|,\qquad \alpha,\beta=1,2,3 \tag{15}$$

Integral (2) over

$$d^{4}x'=d^{3}\mathbf{x}'dt'=d^{3}\mathbf{x}'\frac{dt'}{d\tau}d\tau=d^{3}\mathbf{x}'\dot{f}^{0}(\tau)\,d\tau$$

can be presented in the form

$$\begin{aligned}\delta g^{ik}(x)&=-\kappa\int G_{\mathrm{ret}}(x,x')\,T^{ik}(x')\sqrt{-g_{\mathrm{M}}}d^{4}x'\\&=-\kappa\int\sum_{s}\frac{m_{(s)}\dot{f}_{(s)}^{i}(\tau)\,\dot{f}_{(s)}^{k}(\tau)}{2\pi\,(g_{\mathrm{M}})_{rl}\,\dot{f}_{(s)}^{r}(\tau)\,\dot{f}_{(s)}^{l}(\tau)}\prod_{\alpha=1}^{\alpha=3}\delta_{\alpha}\left(x'^{\alpha}-f_{(s)}^{\alpha}(\tau)\right)\sqrt{-g_{\mathrm{M}}}d^{3}\mathbf{x}'\\&\quad\times\delta\left(\sigma_{\mathrm{M}}(x,f_{(s)}(\tau)\right)\dot{f}_{(s)}^{0}(\tau)\,d\tau\end{aligned} \tag{16}$$

Integration over $\mathbf{x}$ gives

$$\delta g^{ik}(x)=-\kappa\int\sum_{s}\frac{m_{(s)}\dot{f}_{(s)}^{i}(\tau)\,\dot{f}_{(s)}^{k}(\tau)}{2\pi\,(g_{\mathrm{M}})_{rl}\,\dot{f}_{(s)}^{r}(\tau)\,\dot{f}_{(s)}^{l}(\tau)}\sqrt{\frac{g_{\mathrm{M}}}{g_{\mathrm{sp}}}}\delta\left(2\sigma_{\mathrm{M}}(x,f_{(s)}(\tau)\right)\dot{f}_{(s)}^{0}(\tau)\,d\tau \tag{17}$$

Integration over $d\tau$ gives

$$\delta g_{ik}(x)=-\frac{\kappa}{4\pi}\sum_{s}\frac{m_{(s)}\,(g_{\mathrm{M}})_{ij}\,\dot{f}_{(s)}^{j}(\tau_{s})\,(g_{\mathrm{M}})_{kl}\,\dot{f}_{(s)}^{l}(\tau_{s})}{(g_{\mathrm{M}})_{rl}\,\dot{f}_{(s)}^{r}(\tau_{s})\,\dot{f}_{(s)}^{l}(\tau_{s})\left|\frac{d}{d\tau}\sigma_{\mathrm{M}}(x,f(\tau_{s}))\right|}\sqrt{\frac{g_{\mathrm{M}}}{g_{\mathrm{sp}}}}\dot{f}_{(s)}^{0}(\tau_{s}) \tag{18}$$

where $\tau_{s}=\tau_{s}(t,\mathbf{x})$ is a root of the equation

$$2\sigma_{\mathrm{M}}(x,f(\tau_{s}))=(g_{\mathrm{M}})_{ik}\left(x^{i}-f_{(s)}^{i}(\tau_{s})\right)\left(x^{k}-f_{(s)}^{k}(\tau_{s})\right)=0 \tag{19}$$

which can be written in the form

$$\sigma(P,P_{l}')=0 \tag{20}$$

We have

$$\frac{d}{d\tau}\sigma_{\mathrm{M}}\left(x, f\left(\tau_s\right)\right) = -g_{\mathrm{M}ik}\left(x^i - f^i_{(s)}\left(\tau_s\right)\right)\dot{f}^k\left(\tau_s\right) = -\frac{\left(\mathbf{PP}'_l.\mathbf{P}'_l\mathbf{P}'_{l+1}\right)}{d\tau} \tag{21}$$

Using relations (16), (17) one can rewrite the relation (18) in the form

$$\delta g_{ik}\left(x\right) = -\frac{\kappa}{4\pi}\sum_s \frac{m_{(s)}\left(\mathbf{P}'_l\mathbf{P}'_{l+1}.\mathbf{PQ}_i\right)\left(\mathbf{P}'_l\mathbf{P}'_{l+1}.\mathbf{PQ}_k\right)}{\left|\left(\mathbf{PP}'_l.\mathbf{P}'_l\mathbf{P}'_{l+1}\right)\right|\left(\mathbf{P}'_l\mathbf{P}'_{l+1}.\mathbf{P}'_l\mathbf{P}'_{l+1}\right)}\left(\mathbf{P}'_l\mathbf{P}'_{l+1}.\mathbf{PQ}_s\right)\left(g_{\mathrm{M}}\right)^{0s}\sqrt{\frac{g_{\mathrm{M}}}{g_{\mathrm{sp}}}} \tag{22}$$

In the case, when all basic vectors $\mathbf{PQ}_k$ are unite and orthogonal, the determinants g_{M} and g_{sp} are connected by the relation

$$g_{\mathrm{M}} = \det\left|\left|g_{\mathrm{M}ik}\right|\right| = g_{\mathrm{sp}}\left(g_{\mathrm{M}}\right)_{00}, \qquad \left(g_{\mathrm{M}}\right)_{00} = \left(\mathbf{PQ}_0.\mathbf{PQ}_0\right) = \left|\mathbf{PQ}_0\right|^2 \tag{23}$$

Besides

$$\left(\mathbf{P}'_l\mathbf{P}'_{l+1}.\mathbf{PQ}_s\right)\left(g_{\mathrm{M}}\right)^{0s} = \left(\mathbf{P}'_l\mathbf{P}'_{l+1}.\mathbf{PQ}_0\right)\left(g_{\mathrm{M}}\right)^{00} \tag{24}$$

Then the last multipliers of (22) can be written in the form

$$\left(g_{\mathrm{M}}\right)^{00}\sqrt{\frac{g_{\mathrm{M}}}{g_{\mathrm{sp}}}} = \left(\left(g_{\mathrm{M}}\right)_{00}\right)^{-1}\sqrt{\left(g_{\mathrm{M}}\right)_{00}} = \frac{1}{\left|\mathbf{PQ}_0\right|} \tag{25}$$

The constant κ is connected with the gravitational constant G by means of the relation $\kappa = 8\pi G/c^2$. Using (25) and (18), the relation (22) can be rewritten in terms of scalar products

$$\delta g_{ik}\left(P\right) = \delta\left(\left(\mathbf{PQ}_i.\mathbf{PQ}_k\right)\right)$$

$$= -\frac{2G}{c^2}\sum_s m_{(s)}\frac{\theta\left(\left(\mathbf{P}'_l\mathbf{P}.\mathbf{PQ}_0\right)\right)}{\left(\mathbf{P}'_l\mathbf{P}.\mathbf{P}'_l\mathbf{P}'_{l+1}\right)}\frac{\left(\mathbf{P}'_l\mathbf{P}'_{l+1}.\mathbf{PQ}_i\right)\left(\mathbf{P}'_l\mathbf{P}'_{l+1}.\mathbf{PQ}_k\right)\left(\mathbf{P}'_l\mathbf{P}'_{l+1}.\mathbf{PQ}_0\right)}{\left(\mathbf{P}'_l\mathbf{P}'_{l+1}.\mathbf{P}'_l\mathbf{P}'_{l+1}\right)\left|\mathbf{PQ}_0\right|} \tag{26}$$

$$\sigma\left(P, P'_l\right) = 0 \tag{27}$$

where vectors $\mathbf{PQ}_i$, $i = 0, 1, 2, 3$ are basic vectors of the coordinate system at the point P. Vector $\mathbf{PQ}_0$ is timelike. The points P'_l and P'_{l+1} are on the world line $\mathcal{L}_{(s)}$ of sth particle. The points P'_l and P'_{l+1} are separated by infinitesimal distance. All scalar products are taken in the space-time geometry of Minkowski. Besides, one uses the fact, that the metric tensor $g_{ik}\left(P\right)$ at the point P can be presented in the form

$$g_{ik}\left(P\right) = \left(\mathbf{PQ}_i.\mathbf{PQ}_k\right), \qquad i, k = 0, 1, 2, 3 \tag{28}$$

and the scalar product is expressed via the world function by means of the relation (7).

To determine world function σ from relations (26), (27), let us use the relation

$$\left(\mathbf{PS}_1.\mathbf{PS}_2\right) = \sigma\left(P, S_2\right) + \sigma\left(S_1, P\right) - \sigma\left(P, P\right) - \sigma\left(S_1, S_2\right) \tag{29}$$

where S_1 and S_2 are arbitrary points of the space-time. As far as $\sigma\left(P, P\right) = 0$, it may rewritten in the form

$$\sigma\left(S_1, S_2\right) = \sigma\left(P, S_2\right) + \sigma\left(S_1, P\right) - \left(\mathbf{PS}_1.\mathbf{PS}_2\right) \tag{30}$$

Using (8) the relation (30) may be rewritten in terms of scalar products

$$\sigma\left(S_1,S_2\right)=\frac{1}{2}\left(\left(\mathbf{PS}_1.\mathbf{PS}_1\right)+\left(\mathbf{PS}_2.\mathbf{PS}_2\right)-2\left(\mathbf{PS}_1.\mathbf{PS}_2\right)\right) \tag{31}$$

Replacing $Q_i, Q_k,\ i,k\neq 0$ in relation (26) by $S_1,\ S_2$ and substituting in (31), one obtains after transformations

$$\begin{aligned}\delta\sigma\left(S_1,S_2\right)=&-\frac{G}{c^2}\sum_s m_{(s)}\frac{\theta\left(\left(\mathbf{P}_l'\mathbf{P}.\mathbf{PQ}_0\right)\right)\left(\mathbf{P}_l'\mathbf{P}_{l+1}'.\mathbf{PQ}_0\right)}{\left(\mathbf{P}_l'\mathbf{P}.\mathbf{P}_l'\mathbf{P}_{l+1}'\right)\left|\mathbf{PQ}_0\right|}\\&\times\frac{\left(\left(\mathbf{P}_l'\mathbf{P}_{l+1}'.\mathbf{PS}_1\right)-\left(\mathbf{P}_l'\mathbf{P}_{l+1}'.\mathbf{PS}_2\right)\right)^2}{\left(\mathbf{P}_l'\mathbf{P}_{l+1}'.\mathbf{P}_l'\mathbf{P}_{l+1}'\right)}\end{aligned} \tag{32}$$

The relations (32), (27) are completely geometric relations, written in terms of the world function σ_{M} of the Minkowski geometry. According to the deformation principle the relations (32), (27) are valid in any physical space-time geometry (i.e. for any world function σ). It means, that, if the space-time geometry without additional particles is described by the world function σ_0, then appearance of additional particles perturbs the space-time geometry, and it becomes to be described by the world function $\sigma=\sigma_0+\delta\sigma$, where perturbation $\delta\sigma$ of the world function is determined by the relations (32), (27). Scalar products in rhs of (32) should be calculated by means of the world function σ, which is unknown at first. As a result equations (32), (27) form equations for determination of the world function σ.

In the case of continuous distribution of particles the summation in (32) is to be substituted by integration over Lagrangian coordinates $\boldsymbol{\xi}$, labelling the perturbing particles. One obtains

$$\begin{aligned}\delta\sigma\left(S_1,S_2\right)=&-\frac{G}{c^2}\int_V\rho\left(\boldsymbol{\xi}\right)d\boldsymbol{\xi}\frac{\theta\left(\left(\mathbf{P}_l'\mathbf{P}.\mathbf{PQ}_0\right)\right)\left(\mathbf{P}_l'\mathbf{P}_{l+1}'.\mathbf{PQ}_0\right)}{\left(\mathbf{P}_l'\mathbf{P}.\mathbf{P}_l'\mathbf{P}_{l+1}'\right)\left|\mathbf{PQ}_0\right|}\\&\times\frac{\left(\left(\mathbf{P}_l'\mathbf{P}_{l+1}'.\mathbf{PS}_1\right)-\left(\mathbf{P}_l'\mathbf{P}_{l+1}'.\mathbf{PS}_2\right)\right)^2}{\left(\mathbf{P}_l'\mathbf{P}_{l+1}'.\mathbf{P}_l'\mathbf{P}_{l+1}'\right)}\end{aligned} \tag{33}$$

where the total mass M is defined by the relation

$$\int_V\rho\left(\boldsymbol{\xi}\right)d\boldsymbol{\xi}=M \tag{34}$$

The points S_1 and S_2 are arbitrary points of the space-time.

Remark 1

It is possible that equation (1) is valid only for small values of the metric tensor. In this case the relation (33) should be replaced by a set of n equations of the form (33). The first equation determines $\sigma_1=\sigma_0+\delta\sigma_1$, where $\delta\sigma_1$ is defined by the relation (33) with initial σ_0, where ρ is replaced by ρ/n. The second equation determines $\sigma_2=\sigma_1+\delta\sigma_2$, where $\delta\sigma_2$ is defined by the relation (33) with initial σ_1, where ρ is replaced by ρ/n. ... The final nth equation determines $\sigma_n=\sigma_{n-1}+\delta\sigma_n$, where $\delta\sigma_n$ is defined by the relation (33) with initial σ_{n-1}, where ρ is replaced by ρ/n. The exact result $\sigma=\sigma_n$ is obtained at $n\rightarrow\infty$. At any step of solution a change of the world function will be small. As a result one obtains something like a differential equation instead of the finite equation (33)

5. World Function of Non-rotating Body

Let us consider a physical body, which is concentrated in a space volume V. Its density is $\rho(\boldsymbol{\xi})$, where $\boldsymbol{\xi}$ are Lagrangian coordinates of the body points. The body does not rotate. We shall use the inertial coordinate system $x = \{t, \mathbf{x}\} = \{t, x^1, x^2, x^3\}$.

We shall search for solution of equations (33), (27) in the form of a second order polynomial of $(t_1 - t_2)$

$$\sigma(t_1, \mathbf{y}_1; t_2, \mathbf{y}_2) = \frac{1}{2} A(\mathbf{y}_1, \mathbf{y}_2) c^2 (t_2 - t_1)^2 + B(\mathbf{y}_1, \mathbf{y}_2) c (t_2 - t_1) + C(\mathbf{y}_1, \mathbf{y}_2) \qquad (1)$$

$$A(\mathbf{y}_1, \mathbf{y}_2) = 1 - V(\mathbf{y}_1, \mathbf{y}_2), \qquad C(\mathbf{y}_1, \mathbf{y}_2) = -\frac{1}{2}(\mathbf{y}_1 - \mathbf{y}_2)^2 + \delta C(\mathbf{y}_1, \mathbf{y}_2) \qquad (2)$$

where functions A, B and C should be determined as a result of solution of equations (33), (27). In the zeroth order approximation, when the space-time is the space of Minkowski, one has

$$A_0(\mathbf{y}_1, \mathbf{y}_2) = 1, \quad V_0(\mathbf{y}_1, \mathbf{y}_2) = 0, \quad B_0(\mathbf{y}_1, \mathbf{y}_2) = 0, \quad \delta C_0(\mathbf{y}_1, \mathbf{y}_2) = 0 \qquad (3)$$

Let coordinates of points have the form

$$\begin{aligned} P_l' &= \left\{t - \frac{r}{c}, \boldsymbol{\xi}\right\}, \qquad P_{l+1}' = \left\{t - \frac{r}{c} + dT, \boldsymbol{\xi}\right\}, \\ P &= \{t, \mathbf{x}\} \qquad S_1 = \{t_1, \mathbf{y}_1\} \qquad S_2 = \{t_2, \mathbf{y}_2\} \\ Q_0 &= \{t + dt, \mathbf{x}\}, \qquad Q_1 = \left\{t, x^1 + dx^1, x^2, x^3\right\}, \\ Q_2 &= \left\{t, x^1, x^2 + dx^2, x^3\right\}, \qquad Q_3 = \left\{t, x^1, x^2, x^3 + dx^3\right\} \end{aligned} \qquad (4)$$

where coordinates $\boldsymbol{\xi}$ label points of the body. The point P is chosen such, that

$$t = \frac{t_1 + t_2}{2}, \quad \mathbf{x} = \frac{\mathbf{y}_1 + \mathbf{y}_2}{2} \qquad (5)$$

Vectors $\mathbf{PQ}$ in scalar products of the expression (33) are described by coordinates of points P and Q: $\mathbf{PQ} = \{x(P); x(Q)\}$, where $x(P)$ are coordinates of the point P. By means of (4) we have the following coordinates for vectors in (33):

$$\begin{aligned} &\mathbf{P}_l'\mathbf{P} = \left\{t - \frac{r}{c}, \boldsymbol{\xi}; t, \mathbf{x}\right\}, \quad \mathbf{PQ}_0 = \{t, \mathbf{x}; t + dt, \mathbf{x}\}, \quad \mathbf{P}_l'\mathbf{P}_{l+1}' = \left\{t - \frac{r}{c}, \boldsymbol{\xi}; t - \frac{r}{c} + dT, \boldsymbol{\xi}\right\}, \\ &\mathbf{PS}_1 = \{t, \mathbf{x}; t_1, \mathbf{y}_1\}, \quad \mathbf{PS}_2 = \{t, \mathbf{x}; t_2, \mathbf{y}_2\} \end{aligned} \qquad (6)$$

The quantity dT is supposed to be infinitesimal.

In the first approximation the world function has the form

$$\sigma_1(t_1, \mathbf{y}_1; t_2, \mathbf{y}_2) = \frac{1}{2} A_1(\mathbf{y}_1, \mathbf{y}_2) c^2 (t_2 - t_1)^2 + B_1(\mathbf{y}_1, \mathbf{y}_2) c (t_2 - t_1) + C_1(\mathbf{y}_1, \mathbf{y}_2) \qquad (7)$$

As far as $\sigma_1 = \sigma_{\mathrm{M}} + \delta\sigma_1$, it follows from (7)

$$\delta\sigma_1(t_1, \mathbf{y}_1; t_2, \mathbf{y}_2) = -\frac{1}{2} V_1(\mathbf{y}_1, \mathbf{y}_2) c^2 (t_2 - t_1)^2 + B_1(\mathbf{y}_1, \mathbf{y}_2) c (t_2 - t_1) + \delta C_1(\mathbf{y}_1, \mathbf{y}_2) \qquad (8)$$

where

$$\delta C_1(\mathbf{y}_1,\mathbf{y}_2)=C_1(\mathbf{y}_1,\mathbf{y}_2)+\frac{1}{2}(\mathbf{y}_2-\mathbf{y}_1)^2 \tag{9}$$

At $t_2\rightarrow t_1$ the world function $\sigma_1(t_1,\mathbf{y}_1;t_2,\mathbf{y}_1)$ tends to 0. Then it follows from (7)

$$C_1(\mathbf{y}_1,\mathbf{y}_1)=0 \tag{10}$$

Taking into account the symmetry of the world function with respect to transposition $(t_1,\mathbf{y}_1)\leftrightarrow(t_2,\mathbf{y}_1)$, we conclude from (7), that

$$B_1(\mathbf{y}_1,\mathbf{y}_1)=0 \tag{11}$$

According to equation (27), we obtain from (7), (6)

$$\frac{1}{2}|\mathbf{P}_l'\mathbf{P}|^2=\frac{1}{2}A_1(\boldsymbol{\xi},\mathbf{x})c^2\left(\frac{r}{c}\right)^2+B_1(\boldsymbol{\xi},\mathbf{x})r+C_1(\boldsymbol{\xi},\mathbf{x})=0 \tag{12}$$

Solution of (12) has the form

$$\begin{aligned} r &= \frac{-B_1(\boldsymbol{\xi},\mathbf{x})+\sqrt{B_1^2(\boldsymbol{\xi},\mathbf{x})-2C_1(\boldsymbol{\xi},\mathbf{x})A_1(\boldsymbol{\xi},\mathbf{x})}}{A_1(\boldsymbol{\xi},\mathbf{x})}\\ &= -\frac{2C_1(\boldsymbol{\xi},\mathbf{x})}{B_1(\boldsymbol{\xi},\mathbf{x})+\sqrt{B_1^2(\boldsymbol{\xi},\mathbf{x})-2C_1(\boldsymbol{\xi},\mathbf{x})A_1(\boldsymbol{\xi},\mathbf{x})}} \end{aligned} \tag{13}$$

Calculation of other scalar products gives the results

$$|\mathbf{PQ}_0|^2=A_1(\mathbf{y}_1,\mathbf{y}_2)c^2(dt)^2 \tag{14}$$

$$\left(\mathbf{P}_l'\mathbf{P}_{l+1}'.\mathbf{P}_l'\mathbf{P}_{l+1}'\right)=A_1(\boldsymbol{\xi},\boldsymbol{\xi})c^2(dT)^2 \tag{15}$$

$$\begin{aligned} \left(\mathbf{P}_l'\mathbf{P}_{l+1}'.\mathbf{P}_l'\mathbf{P}\right) &= \sigma(P_l',P)+\sigma\left(P_{l+1}',P_l'\right)-0-\sigma\left(P_{l+1}',P\right)\\ &= \sigma\left(P_{l+1}',P_l'\right)-\sigma\left(P_{l+1}',P\right) \end{aligned} \tag{16}$$

$$\begin{aligned} \sigma\left(P_{l+1}',P\right) &= \sigma\left(t-\frac{r}{c}+dT,\boldsymbol{\xi};t,\mathbf{x}\right)\\ &= \frac{1}{2}A_1(\boldsymbol{\xi},\mathbf{x})c^2\left(\frac{r}{c}-dT\right)^2+cB_1(\boldsymbol{\xi},\mathbf{x})\left(\frac{r}{c}-dT\right)+C_1(\boldsymbol{\xi},\mathbf{x}) \end{aligned} \tag{17}$$

$$\begin{aligned} &\left(\mathbf{P}_l'\mathbf{P}_{l+1}'.\mathbf{P}_l'\mathbf{P}\right)=\sigma\left(P_{l+1}',P_l'\right)-\sigma\left(P_{l+1}',P\right)\\ &=\frac{1}{2}A_1(\boldsymbol{\xi},\boldsymbol{\xi})c^2(dT)^2-\left(\frac{1}{2}A_1(\boldsymbol{\xi},\mathbf{x})c^2\left(\frac{r}{c}-dT\right)^2+cB_1(\boldsymbol{\xi},\mathbf{x})\left(\frac{r}{c}-dT\right)+C_1(\boldsymbol{\xi},\mathbf{x})\right)\\ &=-\frac{1}{2}A_1(\boldsymbol{\xi},\mathbf{x})r^2+A_1(\boldsymbol{\xi},\mathbf{x})crdT-B_1(\boldsymbol{\xi},\mathbf{x})r+B_1(\boldsymbol{\xi},\mathbf{x})cdT-C_1(\boldsymbol{\xi},\mathbf{x})+\mathcal{O}\left(dT^2\right) \end{aligned} \tag{18}$$

Taking into account, the relation (12) one obtains

$$\left(\mathbf{P}_l'\mathbf{P}_{l+1}'.\mathbf{P}_l'\mathbf{P}\right)=A_1(\boldsymbol{\xi},\mathbf{x})crdT+B_1(\boldsymbol{\xi},\mathbf{x})cdT+\mathcal{O}\left(dT^2\right) \tag{19}$$

Calculation of $\left(\mathbf{P}'_l\mathbf{P}'_{l+1}.\mathbf{PS}_1\right)-\left(\mathbf{P}'_l\mathbf{P}'_{l+1}.\mathbf{PS}_2\right)$ gives

$$\begin{aligned}
&\left(\mathbf{P}'_l\mathbf{P}'_{l+1}.\mathbf{PS}_1\right)-\left(\mathbf{P}'_l\mathbf{P}'_{l+1}.\mathbf{PS}_2\right)\\
&=\sigma_1\left(P'_l,S_1\right)+\sigma_1\left(P'_{l+1},P\right)-\sigma_1\left(P'_l,P\right)-\sigma_1\left(P'_{l+1},S_1\right)\\
&\quad-\left(\sigma_1\left(P'_l,S_2\right)+\sigma_1\left(P'_{l+1},P\right)-\sigma_1\left(P'_l,P\right)-\sigma_1\left(P'_{l+1},S_2\right)\right)\\
&=\sigma_1\left(P'_l,S_1\right)-\sigma_1\left(P'_{l+1},S_1\right)-\left(\sigma_1\left(P'_l,S_2\right)-\sigma_1\left(P'_{l+1},S_2\right)\right)\\
&=-dT\frac{\partial}{\partial dT}\sigma_1\left(P'_{l+1},S_1\right)+dT\frac{\partial}{\partial dT}\sigma_1\left(P'_{l+1},S_2\right)\\
&=dT\frac{\partial}{\partial dT}\left(\sigma_1\left(P'_{l+1},S_2\right)-\sigma_1\left(P'_{l+1},S_1\right)\right)+\mathcal{O}\left(dT^2\right)
\end{aligned}\tag{20}$$

Using (1) and

$$\mathbf{P}'_l\mathbf{S}_1=\left\{t-\frac{r}{c},\boldsymbol{\xi};t_1,\mathbf{y}_1\right\},\qquad \mathbf{P}'_{l+1}\mathbf{S}_1=\left\{t-\frac{r}{c}+dT,\boldsymbol{\xi};t_1,\mathbf{y}_1\right\}\tag{21}$$

one obtains from (20)

$$\begin{aligned}
&\left(\mathbf{P}'_l\mathbf{P}'_{l+1}.\mathbf{PS}_1\right)-\left(\mathbf{P}'_l\mathbf{P}'_{l+1}.\mathbf{PS}_2\right)\\
&=+A_1\left(\boldsymbol{\xi},\mathbf{y}_1\right)c^2t_1dT-A_1\left(\boldsymbol{\xi},\mathbf{y}_2\right)c^2t_2dT+\left(A_1\left(\boldsymbol{\xi},\mathbf{y}_2\right)-A_1\left(\boldsymbol{\xi},\mathbf{y}_1\right)\right)c^2\left(t-\frac{r}{c}\right)dT\\
&\quad+\left(B_1\left(\boldsymbol{\xi},\mathbf{y}_1\right)-B_1\left(\boldsymbol{\xi},\mathbf{y}_2\right)\right)cdT+\mathcal{O}\left(dT^2\right)
\end{aligned}\tag{22}$$

Let us take into account, that the time coordinate t of the point P has the form (5). The relation (22) takes the form

$$\begin{aligned}
&\left(\mathbf{P}'_l\mathbf{P}'_{l+1}.\mathbf{PS}_1\right)-\left(\mathbf{P}'_l\mathbf{P}'_{l+1}.\mathbf{PS}_2\right)\\
&=+\frac{1}{2}\left(A_1\left(\boldsymbol{\xi},\mathbf{y}_1\right)+A_1\left(\boldsymbol{\xi},\mathbf{y}_2\right)\right)c^2\left(t_1-t_2\right)dT-\left(A_1\left(\boldsymbol{\xi},\mathbf{y}_2\right)-A_1\left(\boldsymbol{\xi},\mathbf{y}_1\right)\right)rcdT\\
&\quad+\left(B_1\left(\boldsymbol{\xi},\mathbf{y}_1\right)-B_1\left(\boldsymbol{\xi},\mathbf{y}_2\right)\right)cdT
\end{aligned}\tag{23}$$

Using (2), the relation (23) can be written in the form

$$\begin{aligned}
&\left(\mathbf{P}'_l\mathbf{P}'_{l+1}.\mathbf{PS}_1\right)-\left(\mathbf{P}'_l\mathbf{P}'_{l+1}.\mathbf{PS}_2\right)\\
&=\left(1-\frac{1}{2}\left(V_1\left(\boldsymbol{\xi},\mathbf{y}_2\right)+V_1\left(\boldsymbol{\xi},\mathbf{y}_1\right)\right)\right)c^2\left(t_1-t_2\right)dT+\left(V_1\left(\boldsymbol{\xi},\mathbf{y}_2\right)-V_1\left(\boldsymbol{\xi},\mathbf{y}_1\right)\right)rcdT\\
&\quad+\left(B_1\left(\boldsymbol{\xi},\mathbf{y}_1\right)-B_1\left(\boldsymbol{\xi},\mathbf{y}_2\right)\right)cdT
\end{aligned}\tag{24}$$

Calculation gives the following result for scalar product $\left(\mathbf{P}'_l\mathbf{P}.\mathbf{PQ}_0\right)$

$$\left(\mathbf{P}'_l\mathbf{P}.\mathbf{PQ}_0\right)=\left(2A_1\left(\boldsymbol{\xi},\mathbf{x}\right)r+B_1\left(\boldsymbol{\xi},\mathbf{x}\right)\right)cdt\tag{25}$$

This scalar product is positive, if r, defined by the relation (13), is positive and $A_1\left(\boldsymbol{\xi},\mathbf{x}\right)>0$.

After substitution of expressions (11), (12), (15), (19) and (24), the expression (33)

takes the form

$$\begin{aligned}
&\delta\sigma\left(S_1,S_2\right)\\
&=-\frac{G}{c^2}\int_V \rho\left(\boldsymbol{\xi}\right)d\boldsymbol{\xi}\frac{\theta\left(\left(\mathbf{P}_l'\mathbf{P}.\mathbf{PQ}_0\right)\right)A_1\left(\boldsymbol{\xi},\mathbf{x}\right)c^2dtdT}{\sqrt{A_1\left(\mathbf{x},\mathbf{x}\right)}cdt}\\
&\times\frac{\left(\begin{array}{c}\left(1-\frac{1}{2}\left(V_1\left(\boldsymbol{\xi},\mathbf{y}_2\right)+V_1\left(\boldsymbol{\xi},\mathbf{y}_1\right)\right)\right)c\left(t_1-t_2\right)\\ +\left(V_1\left(\boldsymbol{\xi},\mathbf{y}_2\right)-V_1\left(\boldsymbol{\xi},\mathbf{y}_1\right)\right)r+\left(B_1\left(\boldsymbol{\xi},\mathbf{y}_1\right)-B_1\left(\boldsymbol{\xi},\mathbf{y}_2\right)\right)\end{array}\right)^2\left(cdT\right)^2}{\left(A_1\left(\boldsymbol{\xi},\mathbf{x}\right)r+B_1\left(\boldsymbol{\xi},\mathbf{x}\right)\right)A_1\left(\boldsymbol{\xi},\boldsymbol{\xi}\right)c^2\left(dT\right)^2cdT}
\end{aligned}\tag{26}$$

After cancellation of multiplier dT and dt, we obtain

$$\begin{aligned}
&\delta\sigma\left(S_1,S_2\right)\\
&=-\frac{G}{c^2}\int_V \rho\left(\boldsymbol{\xi}\right)d\boldsymbol{\xi}\frac{A_1\left(\boldsymbol{\xi},\mathbf{x}\right)}{\sqrt{A_1\left(\mathbf{x},\mathbf{x}\right)}\left(A_1\left(\boldsymbol{\xi},\mathbf{x}\right)r+B_1\left(\boldsymbol{\xi},\mathbf{x}\right)\right)A_1\left(\boldsymbol{\xi},\boldsymbol{\xi}\right)}\\
&\times\left(\begin{array}{c}\left(1-\frac{1}{2}\left(V_1\left(\boldsymbol{\xi},\mathbf{y}_2\right)+V_1\left(\boldsymbol{\xi},\mathbf{y}_1\right)\right)\right)c\left(t_1-t_2\right)\\ +\left(V_1\left(\boldsymbol{\xi},\mathbf{y}_2\right)-V_1\left(\boldsymbol{\xi},\mathbf{y}_1\right)\right)r+\left(B_1\left(\boldsymbol{\xi},\mathbf{y}_1\right)-B_1\left(\boldsymbol{\xi},\mathbf{y}_2\right)\right)\end{array}\right)^2
\end{aligned}\tag{27}$$

where

$$r=\frac{-B_1\left(\boldsymbol{\xi},\mathbf{x}\right)+\sqrt{B_1^2\left(\boldsymbol{\xi},\mathbf{x}\right)-2C_1\left(\boldsymbol{\xi},\mathbf{x}\right)A_1\left(\boldsymbol{\xi},\mathbf{x}\right)}}{A_1\left(\boldsymbol{\xi},\mathbf{x}\right)}\tag{28}$$

One can see, that rhs of (27) is the second order polynomial of (t_1-t_2). Thus, our supposition that the world function is the second order polynomial of (t_1-t_2) is not changed after variation of the world function under influence of additional particles.

$$\delta\sigma_2\left(t_1,\mathbf{y}_1;t_2,\mathbf{y}_2\right)=-\frac{1}{2}V_2\left(\mathbf{y}_1,\mathbf{y}_2\right)c^2\left(t_2-t_1\right)^2+B_2\left(\mathbf{y}_1,\mathbf{y}_2\right)c\left(t_2-t_1\right)+\delta C_2\left(\mathbf{y}_1,\mathbf{y}_2\right)\tag{29}$$

On the other side, it follows from (27)

$$\begin{aligned}
&\delta\sigma_2\left(S_1,S_2\right)\\
&=-\int_V D\left(\mathbf{x},\boldsymbol{\xi}\right)\left(1-\frac{1}{2}\left(V_1\left(\boldsymbol{\xi},\mathbf{y}_2\right)+V_1\left(\boldsymbol{\xi},\mathbf{y}_1\right)\right)\right)^2c^2\left(t_1-t_2\right)^2d\boldsymbol{\xi}\\
&-2\int_V D\left(\mathbf{x},\boldsymbol{\xi}\right)\left(1-\frac{1}{2}\left(V_1\left(\boldsymbol{\xi},\mathbf{y}_2\right)+V_1\left(\boldsymbol{\xi},\mathbf{y}_1\right)\right)\right)c\left(t_1-t_2\right)\\
&\times\left(\left(V_1\left(\boldsymbol{\xi},\mathbf{y}_2\right)-V_1\left(\boldsymbol{\xi},\mathbf{y}_1\right)\right)r+\left(B_1\left(\boldsymbol{\xi},\mathbf{y}_1\right)-B_1\left(\boldsymbol{\xi},\mathbf{y}_2\right)\right)\right)d\boldsymbol{\xi}\\
&-\int_V D\left(\mathbf{x},\boldsymbol{\xi}\right)\left(V_1\left(\boldsymbol{\xi},\mathbf{y}_2\right)-V_1\left(\boldsymbol{\xi},\mathbf{y}_1\right)\right)r+\left(B_1\left(\boldsymbol{\xi},\mathbf{y}_1\right)-B_1\left(\boldsymbol{\xi},\mathbf{y}_2\right)\right)^2d\boldsymbol{\xi}
\end{aligned}\tag{30}$$

where

$$\begin{aligned}
D\left(\mathbf{x},\boldsymbol{\xi}\right)&=\frac{G}{c^2}\frac{\rho\left(\boldsymbol{\xi}\right)A_1\left(\boldsymbol{\xi},\mathbf{x}\right)}{A_1\left(\boldsymbol{\xi},\boldsymbol{\xi}\right)\sqrt{A_1\left(\mathbf{x},\mathbf{x}\right)}\left(A_1\left(\boldsymbol{\xi},\mathbf{x}\right)r+B_1\left(\boldsymbol{\xi},\mathbf{x}\right)\right)}\\
&=\frac{G}{c^2}\frac{\rho\left(\boldsymbol{\xi}\right)A_1\left(\boldsymbol{\xi},\mathbf{x}\right)}{A_1\left(\boldsymbol{\xi},\boldsymbol{\xi}\right)\sqrt{A_1\left(\mathbf{x},\mathbf{x}\right)}\sqrt{B_1^2\left(\boldsymbol{\xi},\mathbf{x}\right)-2C_1\left(\boldsymbol{\xi},\mathbf{x}\right)A_1\left(\boldsymbol{\xi},\mathbf{x}\right)}}
\end{aligned}\tag{31}$$

Here

$$C_1(\boldsymbol{\xi},\mathbf{x}) = -\frac{1}{2}(\mathbf{x}-\boldsymbol{\xi})^2 + \delta C_1(\boldsymbol{\xi},\mathbf{x}) \tag{32}$$

Comparing (29) and (30), one concludes

$$V_2(\mathbf{y}_1,\mathbf{y}_2) = 2\int_V D(\mathbf{x},\boldsymbol{\xi})\left(1-\frac{1}{2}(V_1(\boldsymbol{\xi},\mathbf{y}_2)+V_1(\boldsymbol{\xi},\mathbf{y}_1))\right)^2 d\boldsymbol{\xi} \tag{33}$$

$$B_2(\mathbf{y}_1,\mathbf{y}_2) = -2\int_V D(\mathbf{x},\boldsymbol{\xi})\left(1-\frac{1}{2}(V_1(\boldsymbol{\xi},\mathbf{y}_2)+V_1(\boldsymbol{\xi},\mathbf{y}_1))\right) \times ((V_1(\boldsymbol{\xi},\mathbf{y}_2)-V_1(\boldsymbol{\xi},\mathbf{y}_1))r + (B_1(\boldsymbol{\xi},\mathbf{y}_1)-B_1(\boldsymbol{\xi},\mathbf{y}_2)))\,d\boldsymbol{\xi} \tag{34}$$

$$\delta C_2(\mathbf{y}_1,\mathbf{y}_2) = -\int_V D(\mathbf{x},\boldsymbol{\xi})(V_1(\boldsymbol{\xi},\mathbf{y}_2)-V_1(\boldsymbol{\xi},\mathbf{y}_1))r + (B_1(\boldsymbol{\xi},\mathbf{y}_1)-B_1(\boldsymbol{\xi},\mathbf{y}_2))^2\,d\boldsymbol{\xi} \tag{35}$$

Substituting $V_2, B_2, \delta C_2$ in rhs of equations (33) - (35) instead of $V_1, B_1, \delta C_1$, we obtain the quantities $V_3, B_3, \delta C_3$. Continuing this process, we obtain in the limit, that the quantities $V_n, B_n, \delta C_n$, appear to be equal in both sides of equations (33) - (35). In the developed form these equations are written as follows

$$V(\mathbf{y}_1,\mathbf{y}_2) = \frac{2G}{c^2}\int_V \frac{\rho(\boldsymbol{\xi})A(\boldsymbol{\xi},\mathbf{x})\left(1-\frac{1}{2}(V(\boldsymbol{\xi},\mathbf{y}_2)+V(\boldsymbol{\xi},\mathbf{y}_1))\right)^2}{A(\boldsymbol{\xi},\boldsymbol{\xi})\sqrt{A(\mathbf{x},\mathbf{x})}\sqrt{B^2(\boldsymbol{\xi},\mathbf{x})+A(\boldsymbol{\xi},\mathbf{x})\left((\mathbf{x}-\boldsymbol{\xi})^2-2\delta C(\boldsymbol{\xi},\mathbf{x})\right)}}d\boldsymbol{\xi} \tag{36}$$

$$B(\mathbf{y}_1,\mathbf{y}_2) = -2\frac{G}{c^2}\int_V \frac{\rho(\boldsymbol{\xi})A(\boldsymbol{\xi},\mathbf{x})\left(1-\frac{1}{2}(V(\boldsymbol{\xi},\mathbf{y}_2)+V(\boldsymbol{\xi},\mathbf{y}_1))\right)}{A_1(\boldsymbol{\xi},\boldsymbol{\xi})\sqrt{A(\mathbf{x},\mathbf{x})}\sqrt{B^2(\boldsymbol{\xi},\mathbf{x})+A(\boldsymbol{\xi},\mathbf{x})\left((\mathbf{x}-\boldsymbol{\xi})^2-2\delta C(\boldsymbol{\xi},\mathbf{x})\right)}}d\boldsymbol{\xi} \times ((V(\boldsymbol{\xi},\mathbf{y}_2)-V(\boldsymbol{\xi},\mathbf{y}_1))r + (B(\boldsymbol{\xi},\mathbf{y}_1)-B(\boldsymbol{\xi},\mathbf{y}_2))) \tag{37}$$

$$\delta C(\mathbf{y}_1,\mathbf{y}_2) = -\frac{G}{c^2}\int_V \frac{\rho(\boldsymbol{\xi})A(\boldsymbol{\xi},\mathbf{x})((V(\boldsymbol{\xi},\mathbf{y}_2)-V(\boldsymbol{\xi},\mathbf{y}_1))r + (B(\boldsymbol{\xi},\mathbf{y}_1)-B(\boldsymbol{\xi},\mathbf{y}_2)))^2}{A(\boldsymbol{\xi},\boldsymbol{\xi})\sqrt{A(\mathbf{x},\mathbf{x})}\sqrt{B^2(\boldsymbol{\xi},\mathbf{x})+A(\boldsymbol{\xi},\mathbf{x})\left((\mathbf{x}-\boldsymbol{\xi})^2-2\delta C(\boldsymbol{\xi},\mathbf{x})\right)}}d\boldsymbol{\xi} \tag{38}$$

where

$$\mathbf{x} = \frac{\mathbf{y}_1+\mathbf{y}_2}{2}, \quad A(\mathbf{y}_1,\mathbf{y}_2) = 1 - V(\mathbf{y}_1,\mathbf{y}_2) \tag{39}$$

$$r = \frac{-B(\boldsymbol{\xi},\mathbf{x}) + \sqrt{B^2(\boldsymbol{\xi},\mathbf{x})+A(\boldsymbol{\xi},\mathbf{x})\left((\mathbf{x}-\boldsymbol{\xi})^2-2\delta C(\boldsymbol{\xi},\mathbf{x})\right)}}{A(\boldsymbol{\xi},\mathbf{x})} \tag{40}$$

It follows from (37) - (38), that for $\mathbf{y}_1 = \mathbf{y}_2 = \mathbf{x}$

$$B(\mathbf{x},\mathbf{x}) = 0, \qquad \delta C(\mathbf{x},\mathbf{x}) = 0 \tag{41}$$

Equations (36) - (38) are three integral equations for determination of three quantities $V(\mathbf{y}_1,\mathbf{y}_2), B(\mathbf{y}_1,\mathbf{y}_2), \delta C(\mathbf{y}_1,\mathbf{y}_2)$, which determine the world function

$$\sigma(t_1,\mathbf{y}_1;t_2,\mathbf{y}_2) = \frac{1}{2}c^2(t_2-t_1)^2 - \frac{1}{2}(\mathbf{y}_1-\mathbf{y}_2)^2 - \frac{1}{2}V(\mathbf{y}_1,\mathbf{y}_2)c^2(t_2-t_1)^2 + B(\mathbf{y}_1,\mathbf{y}_2)c(t_2-t_1) + \delta C_1(\mathbf{y}_1,\mathbf{y}_2) \tag{42}$$

6. Dynamic Equations for World Function, Generated by Non-rotating Sphere

Let the shape of the physical body be a sphere of radius R. Let us introduce parameter $\varepsilon = r_g/R$, where $r_g = 2GM/c^2$ is so called gravitational radius. Let

$$\varepsilon = \frac{2G}{c^2}\int_V \frac{\rho(\boldsymbol{\xi})}{R} d\boldsymbol{\xi} \ll 1 \tag{1}$$

Then it follows from equations (36) - (38), that

$$V(\mathbf{y}_1, \mathbf{y}_2) = \mathcal{O}(\varepsilon), \qquad B(\mathbf{y}_1, \mathbf{y}_2) = \mathcal{O}\left(\varepsilon^2\right), \qquad \delta C(\mathbf{y}_1, \mathbf{y}_2) = \mathcal{O}\left(\varepsilon^3\right) \tag{2}$$

If $\varepsilon \ll 1$, equations (36) - (38) can be solved by means of successive approximations. In the first approximation one obtains

$$V_1(\mathbf{y}_1, \mathbf{y}_2) = \frac{2G}{c^2}\int_V \frac{\rho(\boldsymbol{\xi})}{\sqrt{\left(\frac{|\mathbf{y}_1+\mathbf{y}_2|^2}{4} - \boldsymbol{\xi}\right)^2}} d\boldsymbol{\xi} + \mathcal{O}\left(\varepsilon^2\right) \tag{3}$$

$$B_1(\mathbf{y}_1, \mathbf{y}_2) = 0, \quad \delta C_1(\mathbf{y}_1, \mathbf{y}_2) = 0 \tag{4}$$

If $\rho(\boldsymbol{\xi})$

$$\rho(\boldsymbol{\xi}) = \begin{cases} \rho_0 \text{ if } |\boldsymbol{\xi}| < R \\ 0 \text{ if } |\boldsymbol{\xi}| > R \end{cases}, \qquad \rho_0 = \frac{3M}{4\pi R^3} = \text{const} \tag{5}$$

where M is the sphere mass, then

$$V_1(\mathbf{y}_1, \mathbf{y}_2) = \begin{cases} \frac{2GM}{c^2|\mathbf{x}|} & \text{if } |\mathbf{x}| > R \\ 3\frac{GM}{c^2R} - \frac{GM}{c^2R^3}|\mathbf{x}|^2 & \text{if } |\mathbf{x}| < R \end{cases}, \qquad \mathbf{x} = \frac{\mathbf{y}_1 + \mathbf{y}_2}{2} \tag{6}$$

In the second approximation one obtains

$$V_2(\mathbf{y}_1, \mathbf{y}_2) = \frac{2G}{c^2}\int_V \frac{\rho_0(\boldsymbol{\xi})\sqrt{A_1(\boldsymbol{\xi}, \mathbf{x})}\left(1 - \frac{1}{2}(V_1(\boldsymbol{\xi}, \mathbf{y}_2) + V_1(\boldsymbol{\xi}, \mathbf{y}_1))\right)^2}{A_1(\boldsymbol{\xi}, \boldsymbol{\xi})\sqrt{A_1(\mathbf{x}, \mathbf{x})}\sqrt{(\mathbf{x} - \boldsymbol{\xi})^2}} d\boldsymbol{\xi} + \mathcal{O}\left(\varepsilon^3\right) \tag{7}$$

$$\begin{aligned} B_2(\mathbf{y}_1, \mathbf{y}_2) = & -2\frac{G}{c^2}\int_V \frac{\rho_0(\boldsymbol{\xi})\sqrt{A_1(\boldsymbol{\xi}, \mathbf{x})}\left(1 - \frac{1}{2}(V_1(\boldsymbol{\xi}, \mathbf{y}_2) + V_1(\boldsymbol{\xi}, \mathbf{y}_1))\right)}{A_1(\boldsymbol{\xi}, \boldsymbol{\xi})\sqrt{A_1(\mathbf{x}, \mathbf{x})}\sqrt{(\mathbf{x} - \boldsymbol{\xi})^2}} d\boldsymbol{\xi} \\ & \times (V_1(\boldsymbol{\xi}, \mathbf{y}_2) - V_1(\boldsymbol{\xi}, \mathbf{y}_1))\, r \end{aligned} \tag{8}$$

where

$$r = \frac{\sqrt{(\mathbf{x} - \boldsymbol{\xi})^2}}{\sqrt{A_1(\boldsymbol{\xi}, \mathbf{x})}}$$

$$B_2(\mathbf{y}_1,\mathbf{y}_2) = -2\frac{G}{c^2}\int_V \rho_0(\boldsymbol{\xi})\left(V_1(\boldsymbol{\xi},\mathbf{y}_2) - V_1(\boldsymbol{\xi},\mathbf{y}_1)\right)d\boldsymbol{\xi} + \mathcal{O}\left(\varepsilon^3\right) \tag{9}$$

$$\begin{aligned}\delta C_2(\mathbf{y}_1,\mathbf{y}_2) &= -\frac{G}{c^2}\int_V \frac{\rho_0(\boldsymbol{\xi})\sqrt{A_1(\boldsymbol{\xi},\mathbf{x})}\left(\left(V_1(\boldsymbol{\xi},\mathbf{y}_2) - V_1(\boldsymbol{\xi},\mathbf{y}_1)\right)\frac{\sqrt{(\mathbf{x}-\boldsymbol{\xi})^2}}{\sqrt{A_1(\boldsymbol{\xi},\mathbf{x})}}\right)^2}{A_1(\boldsymbol{\xi},\boldsymbol{\xi})\sqrt{A_1(\mathbf{x},\mathbf{x})}\sqrt{\left((\mathbf{x}-\boldsymbol{\xi})^2\right)}}d\boldsymbol{\xi} \\ &= -\frac{G}{c^2}\int_V \frac{\rho_0(\boldsymbol{\xi})\sqrt{(\mathbf{x}-\boldsymbol{\xi})^2}\left(V_1(\boldsymbol{\xi},\mathbf{y}_2) - V_1(\boldsymbol{\xi},\mathbf{y}_1)\right)^2}{A_1(\boldsymbol{\xi},\boldsymbol{\xi})\sqrt{A_1(\mathbf{x},\mathbf{x})A_1(\boldsymbol{\xi},\mathbf{x})}}d\boldsymbol{\xi} = \mathcal{O}\left(\varepsilon^3\right)\end{aligned} \tag{10}$$

We obtain

$$V_2(\mathbf{y}_1,\mathbf{y}_2) = \frac{2G}{c^2}\int_V \frac{\rho_0(\boldsymbol{\xi})\sqrt{A_1(\boldsymbol{\xi},\mathbf{x})}\left(1 - \frac{1}{2}\left(V_1(\boldsymbol{\xi},\mathbf{y}_2) + V_1(\boldsymbol{\xi},\mathbf{y}_1)\right)\right)^2}{A_1(\boldsymbol{\xi},\boldsymbol{\xi})\sqrt{A_1(\mathbf{x},\mathbf{x})}\sqrt{(\mathbf{x}-\boldsymbol{\xi})^2}}d\boldsymbol{\xi} + \mathcal{O}\left(\varepsilon^3\right) \tag{11}$$

$$\begin{aligned}V_2(\mathbf{y}_1,\mathbf{y}_2) &= V_1(\mathbf{y}_1,\mathbf{y}_2) + \frac{G}{c^2}\int_V \frac{\rho_0(\boldsymbol{\xi})\left(-V_1(\boldsymbol{\xi},\mathbf{x}) + 2V_1(\boldsymbol{\xi},\boldsymbol{\xi}) + V_1(\mathbf{x},\mathbf{x})\right)}{\sqrt{(\mathbf{x}-\boldsymbol{\xi})^2}}d\boldsymbol{\xi} \\ &\quad -\frac{G}{c^2}\int_V \frac{\rho_0(\boldsymbol{\xi})\left(V_1(\boldsymbol{\xi},\mathbf{y}_2) + V_1(\boldsymbol{\xi},\mathbf{y}_1)\right)}{\sqrt{(\mathbf{x}-\boldsymbol{\xi})^2}}d\boldsymbol{\xi} + \mathcal{O}\left(\varepsilon^3\right)\end{aligned} \tag{12}$$

$$\begin{aligned}V_2(\mathbf{y}_1,\mathbf{y}_2) &= V_1(\mathbf{y}_1,\mathbf{y}_2) + \frac{2G}{c^2}\int_V \frac{\rho_0(\boldsymbol{\xi})V_1(\boldsymbol{\xi},\boldsymbol{\xi})}{\sqrt{(\mathbf{x}-\boldsymbol{\xi})^2}}d\boldsymbol{\xi} + \mathcal{O}\left(\varepsilon^3\right) \\ &\quad +\frac{G}{c^2}\int_V \frac{\rho_0(\boldsymbol{\xi})\left(-V_1(\boldsymbol{\xi},\mathbf{x}) + V_1(\mathbf{x},\mathbf{x}) - V_1(\boldsymbol{\xi},\mathbf{y}_2) - V_1(\boldsymbol{\xi},\mathbf{y}_1)\right)}{\sqrt{(\mathbf{x}-\boldsymbol{\xi})^2}}d\boldsymbol{\xi}\end{aligned} \tag{13}$$

Estimation of (13) in the case, when $|\mathbf{y}_1|, |\mathbf{y}_2|, |\mathbf{x}| \gg R$, has the form

$$V_2(\mathbf{y}_1,\mathbf{y}_2) = V_1(\mathbf{y}_1,\mathbf{y}_2) + \frac{6}{5}\varepsilon^2\frac{R}{|\mathbf{x}|} - \frac{\varepsilon^2}{2}\frac{R^2}{|\mathbf{x}|^2}\left(1 + \frac{2|\mathbf{x}|}{|\mathbf{y}_1|} + \frac{2|\mathbf{x}|}{|\mathbf{y}_2|}\right) + \mathcal{O}\left(\varepsilon^3\right) \tag{14}$$

where $V_1(\mathbf{y}_1,\mathbf{y}_2)$ is determined by the relation (6), and

$$\varepsilon = \frac{2GM}{c^2R} \ll 1 \tag{15}$$

In the case, when $\mathbf{y}_1 = \mathbf{y}_2 = \mathbf{x}$, we have

$$V_2(\mathbf{x},\mathbf{x}) = V_1(\mathbf{x},\mathbf{x}) + \frac{6}{5}\varepsilon^2\frac{R}{|\mathbf{x}|} - \frac{5}{2}\varepsilon^2\frac{R^2}{|\mathbf{x}|^2} + \mathcal{O}\left(\varepsilon^3\right) \tag{16}$$

We obtain for the quantity $B_2(\mathbf{y}_1,\mathbf{y}_2)$ for $|\mathbf{y}_2|, |\mathbf{y}_1| \gg R$

$$\begin{aligned}B_2(\mathbf{y}_1,\mathbf{y}_2) &= -2\frac{G}{c^2}\int_V \rho_0(\boldsymbol{\xi})\left(V_1(\boldsymbol{\xi},\mathbf{y}_2) - V_1(\boldsymbol{\xi},\mathbf{y}_1)\right)d\boldsymbol{\xi} + \mathcal{O}\left(\varepsilon^3\right) \\ &= -2\frac{GM}{c^2}\left(V_1(0,\mathbf{y}_2) - V_1(0,\mathbf{y}_1)\right) + \mathcal{O}\left(\varepsilon^3\right) \\ &= -\varepsilon^2R^2\left(\frac{1}{|\mathbf{y}_2|} - \frac{1}{|\mathbf{y}_1|}\right) + \mathcal{O}\left(\varepsilon^3\right)\end{aligned} \tag{17}$$

$$B_2(\mathbf{x},\mathbf{x}) = 0 \tag{18}$$

Thus, for small $\varepsilon = 2GM/(Rc^2)$ and $|\mathbf{x}| \gg R$, the calculated value of metric tensor, determined by the quantities $V_1(\mathbf{y}_1,\mathbf{y}_1), B_1(\mathbf{y}_1,\mathbf{y}_1), \delta C_1(\mathbf{y}_1,\mathbf{y}_1)$ coincides with the metric tensor, calculated in Newtonian approximation of the general relativity.

At large values of parameter ε the quantity $V(\mathbf{x},\mathbf{x})$ remains to be less, than unity. Indeed, setting $\mathbf{y}_1 = \mathbf{y}_2 = \mathbf{x}$ in exact equations (36) - (38), we obtain

$$V(\mathbf{x},\mathbf{x}) = \frac{2G}{c^2}\int_V \frac{\rho(\boldsymbol{\xi}) A(\boldsymbol{\xi},\mathbf{x})\left(1-\frac{1}{2}\left(V(\boldsymbol{\xi},\mathbf{x})+V(\boldsymbol{\xi},\mathbf{x})\right)\right)^2}{A(\boldsymbol{\xi},\boldsymbol{\xi})\sqrt{A(\mathbf{x},\mathbf{x})}\sqrt{B^2(\boldsymbol{\xi},\mathbf{x})+A(\boldsymbol{\xi},\mathbf{x})\left((\mathbf{x}-\boldsymbol{\xi})^2-2\delta C(\boldsymbol{\xi},\mathbf{x})\right)}}d\boldsymbol{\xi}$$

$$B(\mathbf{x},\mathbf{x}) = 0, \qquad \delta C(\mathbf{x},\mathbf{x}) = 0 \tag{19}$$

Rewriting equation (19) in the form

$$V(\mathbf{x},\mathbf{x})\sqrt{1-V(\mathbf{x},\mathbf{x})}$$

$$= \frac{2G}{c^2}\int_V \frac{\rho(\boldsymbol{\xi}) A(\boldsymbol{\xi},\mathbf{x})\left(1-\frac{1}{2}\left(V(\boldsymbol{\xi},\mathbf{x})+V(\boldsymbol{\xi},\mathbf{x})\right)\right)^2}{A(\boldsymbol{\xi},\boldsymbol{\xi})\sqrt{B^2(\boldsymbol{\xi},\mathbf{x})+A(\boldsymbol{\xi},\mathbf{x})(\mathbf{x}-\boldsymbol{\xi})^2-2A(\boldsymbol{\xi},\mathbf{x})\delta C(\boldsymbol{\xi},\mathbf{x})}}d\boldsymbol{\xi} \tag{20}$$

we conclude, that equation (20) contains only solutions with $V(\mathbf{x},\mathbf{x}) \le 1$. In other words, component $g_{00} = c^2(1-V(\mathbf{x},\mathbf{x}))$ of the metric tensor cannot change its sign. *It means that non-rotating physical body of any size and of any mass cannot generate a black hole.*

This result disagrees with the result of general relativity, but it agrees with the common sense. To obtain the reason of such unexpected result, we calculate the quantities $A, B, \delta C$ inside the uniform heavy sphere of radius R and mass M. At calculation we suppose that the quantity

$$\varepsilon = \frac{r_g}{R} = \frac{2GM}{c^2R} \ll 1 \tag{21}$$

where r_g is the gravitational radius of the sphere.

For $|\mathbf{x}| < R$ results of calculations looks as follows (Details of calculations are rather bulky, and we omit them)

$$V_2(\mathbf{x},\mathbf{x}) = \varepsilon\left(\frac{3}{2}-\frac{1}{2}\frac{\mathbf{x}^2}{R^2}\right) - \varepsilon^2\frac{153}{64} + \varepsilon^2\frac{37}{32}\frac{\mathbf{x}^2}{R^2} - \varepsilon^2\frac{61}{320}\frac{|\mathbf{x}|^4}{R^4} + \mathcal{O}\left(\varepsilon^3\right) \tag{22}$$

The gravitational force inside the region $|\mathbf{x}| < R$ has the form

$$\mathbf{F} = \boldsymbol{\nabla} V_2(\mathbf{x},\mathbf{x}) = -\frac{\varepsilon}{R^2}\mathbf{x} + \frac{\varepsilon^2}{R^2}\frac{37}{16}\mathbf{x} - \frac{61}{80}\frac{\varepsilon^2}{R^2}\frac{|\mathbf{x}|^2}{R^2}\mathbf{x}, \qquad |\mathbf{x}| < R \tag{23}$$

It follows from (23), that if $\varepsilon > \frac{16}{37} \approx 0.43$, the region, where the gravitational force is directed from the center, appears near the point $\mathbf{x} = 0$. If $\varepsilon \ge 0.65$, the gravitational force is directed from the center of the sphere in the whole region $|\mathbf{x}| < R$.

Thus, inside the heavy sphere the regions of antigravitation may appear at large values of ε. To understand this unexpected circumstance, let us note, that dynamical

(not completely relativistic) approach and geometrical (completely relativistic) approach to gravitational phenomena disagree in some points.

The Newtonian gravitational potential of a uniform heavy sphere of radius R has the form

$$\varphi(\mathbf{x}) = \begin{cases} \frac{GM}{|\mathbf{x}|} & \text{if } |\mathbf{x}| > R \\ \frac{3GM}{2R} - \frac{GM}{2R^3}|\mathbf{x}|^2 & \text{if } |\mathbf{x}| < R \end{cases} \tag{24}$$

where M is the of the sphere. The gravitational potential φ is maximal at the point $\mathbf{x} = 0$, whereas the gravitational force $\mathbf{F} = \boldsymbol{\nabla}\varphi$ is minimal at the point $\mathbf{x} = 0$ ($\mathbf{F} = 0$ at $\mathbf{x} = 0$). The space-time geometry is connected with the gravitational potential $g_{00} = (c^2 - 2\varphi)$, but not with the gravitational force $\mathbf{F}$.

Gravitational potential φ inside the hallow sphere of mass M is proportional to the mass M, but φ =const, and the gravitational force $\mathbf{F} = 0$ inside the sphere. From dynamic (differential) viewpoint this fact is explained as follows. Gravitational influence of different parts of the hallow sphere compensate inside the sphere. If the gravitational law distinguishes from the Newtonian one, such a compensation may disappear, and an induced antigravitation may appear, because the attraction force, generated by any part of the hallow sphere, is directed from the center of the sphere.

From the geometric (integral) viewpoint an appearance of the induced antigravitation regions is natural, because the gravitational potential increases in such regions with increase of amount of the matter. As to the gravitational force, it may have any direction.

7. Concluding Remarks

Thus, the extended general relativity (EGR) is the next stage of the physics geometrization. At this stage we have the monistic conception, containing only one fundamental quantity: world function σ. The gravitational field, which is one of fundamental quantities of the general relativity (GR), is now only an attribute of the world function. From viewpoint of extended general relativity (EGR) the gravitational field is not a physical essence. It is only a manner of the particle interaction description. In particular, from viewpoint of EGR the gravitational field cannot exist separate from the matter. Such a change of approach to the gravitational field is connected with a usage of the relativistic concept of the events nearness.

Any monistic conception is a result of development of the preceding pluralistic conception, and the monistic conception is more perfect as a rule, than the preceding pluralistic one. The extended general relativity (EGR) is obtained as a result of overcoming of defects of the general relativity (GR): (1) usage of only inconsistent Riemannian space-time geometry, (2) use of inadequate (nonrelativistic) concepts and quantities. EGR is to be considered as a more perfect conception, than GR. Results obtained in the framework of EGR are more dependable, than results, obtained in the framework of GR. In particular, conclusion on impossibility of the dark hole existence in EGR is more dependable, than existence of the black holes in the framework of GR. Besides, impossibility of the gravi-

tational collapsing, leading to a formation of a black hole, is confirmed by appearance of induced antigravitation in EGR.

Besides, the mathematical technique of EGR is the same for all (continuous and discrete) geometries. Dynamic equations for the world function are written in the coordinateless form. This circumstance admits one to eliminate consideration of any coordinate transformation.

There is a possibility, that some problems of contemporary cosmology (dark matter, dark energy) are a result of imperfect theory of gravitation. More correct results of EGR, concerning dark holes, admit to hope, that EGR will be able to solve difficult problems of contemporary cosmology.

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Spatially Homogeneous String Cosmological Models with Bulk Viscosity in $f(R,T)$ Gravity Theory

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Abstract: We discuss the field equations in f(R,T) gravity theory for the general class of Bianchi models filled with bulk viscous fluid together with a one-dimensional cosmic strings. We obtain two classes of cosmological solutions of the field equation by setting the average scale factor and one of the scale factors of the model proportional to a power function of the spatial volume. One class of solution represents on accelerated expanding universe having in the infinite past singularity. The other class of solution also represents on accelerated expanding having singularity at the initial time. The physical and kinematical properties of the models are discussed. We observe that the models of the two classes are compatible with the results of recent observations.

Keywords: Bianchi Spaces; Bulk Viscosity; $f(R,T)$ Gravity Theory
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1. Introduction

The present day universe is satisfactorily described by spatially homogeneous and isotropic perfect fluid Friedmann-Robertson-Walker(FRW)space-times. The universe in a smaller scale is neither homogeneous nor isotropic nor we do expect the universe to have these properties in its early stages of evolution. Spatially homogeneous and anisotropic cosmological models have been widely studied in the framework of general relativity and alternative theories of gravitation in the search of realistic picture of the universe in its early stages of evolution. Bianchi spaces are useful tools for constructing such models.

The evolution of isotropic cosmological models in the presence of perfect fluid has

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been extensively studied by many cosmologists. It has been certainly of interest to study cosmologies with richer structure both geometrically and physically than the standard perfect fluid models. It is of interest to take into account dissipation processes such as viscosity in cosmological models. Viscous fluid models has been used in an attempt to explain the observed highly isotropic matter distribution on the high entropy per baryon in the present state of the universe. Misner [1-2] suggested that the strong dissipation due to neutrino viscosity may considerably reduce the anisotropy of black- body radiation. The viscosity mechanism in cosmology can explain the anomalously high entropy per baryon in the present universe. A uniform cosmological model filled with fluid which possesses pressure and bulk viscosity was investigated by Murphy [3] and has shown that the big-bang type singularity appears in the infinite past. Of late, there have been considerable interests in cosmological models with bulk viscosity, since bulk viscosity leads to the accelerated expansion phase of the early universe, popularly known as the inflationary phase. The possibility of bulk viscosity leading to inflationary -like solutions in general relativistic FRW models has been investigated by several authors viz. Barrow [4]; Padmanabhan and Chitre [5]; Martens [6] etc. .

In recent years there has been considerable interest in string cosmology as cosmic strings play an important role in the study of the early universe.These strings arise during the phase transition after the big-bang explosion as the temperature goes down below some critical temperature as predicted by grand unified theories. Moreover, it is expected that topological defects could have formed naturally during the phase transition followed by spontaneous broken symmetries. Cosmic strings, being linear topological defects, have very interesting properties and might play important roles in the structure formation (Kibble, Everett and Vilenkin) [7-9]. Letelier[10] obtained cosmological solutions of cloud formed by massive strings with particles attached along its extension in Bianchi type-I and Kantowski-Sachs space times. The cloud of strings is the generalization of Takabayasi 's realistic model of string called p-string Letelier [11]. Ram and Singh [12] obtained exact Bianchi type-III cosmological solutions of massive strings in the presence of magnetic field .Bali and Singh [13], Bali and Pradhan [14], Pradhan and Chouhan [15] etc. have investigated Bianchi type string cosmological models in general relativity. Reddy [16], Reddy et al [17] Reddy and Naidu [18] have studied string cosmological models in different contexts. Several authors viz. Chodos and Detwater [19], Youshimura [20], Chatterjee [21], Krori et al. [22], Rahaman et al [23], Mohanty et al.[24] etc. constructed higher dimensional string cosmological models in certain theories of gravitation.

Recent most remarkable observational discoveries have shown that our current universe is not only only expanding but also accelerating. This was first observed from high red shift supernova Ia Riess et al.[25-26], Perlmutter [27], Astier et al. [28], and confirmed later by cross checks from the cosmic microwave background radiation Bennett et al.[29] , Spergel et al. [30-31] and large scale structure Tegmark et al. [32]. In Einstein,s general relativity, in order to have such acceleration, one needs to introduce a component to the matter distribution of the universe with a large negative pressure. This component is usually referred as dark energy (DE) . Astronomical observations indicate that our universe

is flat and currently consists of approximately $\frac{2}{3}$ dark energy and $\frac{1}{3}$ dark matter source of dark energy. Many radically different models have been proposed such as a tiny positive cosmological constant, quintessence(Caldwell et al. [33], Liddle and Scherrer [34], Steinhardt et al. [35]. In view of the late time acceleration of the universe and the existence of dark energy and dark matter, several modified theories of gravity have been developed and studied.In particular, f(R) gravity theory studied by Capozziello et al. [36], Nojiri and Odintsov [37], Carroll et al.[38] is a modified gravity model constructed by replacing the gravitational Lagrangian with a general function of the Ricci scalar R . Modified f(R) gravity theory gives an easy unification of early time inflation and late time acceleration Shamir [39]. Very recently, Singh and Singh [40] investigated functional form of f(R) for a known scale factor in anisotropic locally-rotationally -symmetric (LRS) Bianchi I model with perfect fluid as a source of matter. By assuming the deceleration parameter to be a constant and the shear scalar proportional to the expansion scalar they presented a power -law form of scale factors. A further generalization of f(R) gravity theory has been proposed by Harko et al. [41],is known as f(R,T) theory of gravity where as usual R is the Ricci scalar and T is the trace of the energy -momentum tensor. They also argued that due to the coupling of the matter and geometry, this gravity model depends on a source term, which is nothing but the variation of the matter stress-energy tensor. They have also demonstrated that the possible reconstruction of arbitrary FRW cosmologies by an appropriate choice of a function f(T) . In this theory the covariant divergence of the stress energy tensor is non-zero. As a result, the motion of test particles is not along geodesic path due to presence of an extra force perpendicular to the four velocity. The cosmic acceleration in the modified f(R,T) theory results not only from geometrical contribution but also from the matter content. Subsequently, Houndjo [42] has chosen $f(R,T) = f_1(R) + f_2(T)$ and discussed transition of matter dominated era to an accelerated phase. Sharif et al.[43] have studied thermodynamics in this f(R, T) theory and Azizi [44] examined the possibility of wormhole geometry in f(R, T) gravity. The f(R, T) gravity models can explain the late time cosmic accelerated expansion of the universe. Several authors viz., Adhav, [45], Sharif and Zubair [46], Reddy et al.[47-49], Chaubey and Shukla [50], Ram et al.[51] have investigated spatially homogeneous cosmological models in f(R, T) theory of gravity in the presence of perfect fluid. Ram and Priyanka [52] presented Kaluza-Klein cosmological models in the presence of perfect fluid in f(R, T) gravity. Reddy et al. [53] have considered a locally rotationally symmetric (LRS) Bianchi type-II space -time with cosmic strings and bulk viscosity in a modified theory of gravity . Very recently, Ahmed and Pradhan [54] have studied a cosmological model in f(R, T) gravity of Bianchi type V by assuming $f(R, T) = f_1(R) + f_2(T)$. Chakraborty [55] formulated an alternative f(R, T) gravity theory and dark energy model.

Motivated by the above works, in this paper, we investigate a new class of Bianchi models in the presence of cosmic strings and bulk viscous fluid within the frame work of f(R, T) gravity theory. By using $f(R, T) = f_1(R) + f_2(T)$, which was proposed by Harko et al. (2011), the field equations of the metric in f(R, T) gravity are given in Sect.2. In Sect.3 we obtain exact solutions of the field equations on assuming the barotropic equa-

tion of state . We also discuss some physical and kinematical properties of the models. Conclusion in last in in Sect.4.

2. Models and Field Equations

The f(R,T) gravity theory is the modification of general relativity. In the of f(R,T) gravity depending on the native of the matter source, we obtain several theoretical models corresponding to different matter contributions for f(R,T) gravity are possible. However Harko et al. (2011) gave three classes of these models

$$f(R,T) = \begin{cases} R + 2f(T) \\ f_1(R) + f_2(T) \\ f_1(R) + f_2(R)f_3(T) \end{cases} \tag{1}$$

In this paper, we are focussed to the second class i.e.

$$f(R,T) = f_1(R) + f_2(T) \tag{2}$$

with, $f_1(R) = \mu_1 R$ and $f_2(T) = \mu_2 T$, μ_1 and μ_2 are arbitrary parameters.

We set $\mu_1 = \mu_2 = \mu$, as discussed by Ahmed and Pradhan (2013) so that

$$f(R,T) = \mu(R + T) \tag{3}$$

With this choice of f(R,T). The field equations in f(R,T) gravity theory are Harko et al. (2011)

$$\mu R_{ij} - \frac{1}{2}\mu(R+T)g_{ij} + (g_{ij}\Box - \nabla_i\nabla_j)\mu = 8\pi T_{ij} - \mu T_{ij} + \mu(2T_{ij} + pg_{ij}) \tag{4}$$

Setting $(g_{ij} - \nabla_i\nabla_j)\mu$=0 (Ahmed and Pradhan 2013), we obtain

$$\mu G_{ij} = 8\pi T_{ij} + \mu T_{ij} + (\mu p + \frac{1}{2}\mu T)g_{ij} \tag{5}$$

where $G_{ij} = R_{ij} - \frac{1}{2}Rg_{ij}$ is the Einstein's tensor . This could be rearranged as

$$G_{ij} - \left(p + \frac{1}{2}T\right)g_{ij} = \left(\frac{8\pi + \mu}{\mu}\right)T_{ij} \tag{6}$$

The diagonal form of the metric of general class of Bianchi cosmological model is given by

$$ds^2 = dt^2 - a_1^2dx^2 - a_2^2e^{-2x}dy^2 - a_3^2e^{-2mx}dz^2 \tag{7}$$

where $a_1(t)$, $a_2(t)$ and $a_3(t)$ are cosmic scale factors We have different Bianchi models as follows: type III corresponds to $m = 1$, type VI_0 corresponds to $m = 1$, type VI_0 corresponds to $m = -1$, and all other in given VI_h, where in $m = h - 1$

We consider the energy-momentum tensor of a bulk viscous fluid containing one dimensional cosmic strings as (Reddy et al. 2013):

$$T_{ij} = (\rho + \bar{p})u_i u_j + \bar{p}g_{ij} - \lambda x_i x_j \tag{8}$$

where the bulk viscous pressure $\bar{p}$ is given by

$$\bar{p} = p - 3\zeta H \tag{9}$$

Here ρ is the rest energy density of the system ζ is the coefficient of bulk viscosity $\bar{p}$ is the bulk viscous pressure, H is Hubble parameter and λ is the string tension density. Also u^i is the four velocity vector and x^i is unit-space-like vector representing the direction of the strings which satisfy

$$u^i u_i = -x^i x_i = 1, \quad u^i x_i = 0 \tag{10}$$

where

$$u^i = (0,0,0,1), \quad x^i = (\frac{1}{a},0,0,0). \tag{11}$$

The proper energy density can be expressed as

$$\rho = \rho_p + \lambda \tag{12}$$

where ρ_p is the rest density of the particles attached to the strings Letelier [10], Khadekar and Tade [56], Pradhan et al. [56]. The tension density λ may be positive or negative. A more general relationship between the proper rest energy density ρ and string tension density λ may be taken of the form

$$\rho = r\lambda \tag{13}$$

where r is an arbitrary constant which can take both positive and negative values. The negative values of r lead to the absence of strings in the universe and the positive values show the presence of one dimensional strings in the cosmic fluid.Therefore, the energy density of the particles attached to the strings is

$$\rho_p = \rho - \lambda = (r-1)\lambda \tag{14}$$

Also, for a barotropic fluid, the combined effect of the proper pressure and the bulk viscous pressure can be expressed as

$$\bar{p} = p - 3\zeta H = \zeta\rho \tag{15}$$

where

$$\xi = \xi_0 - \tau \quad (0 \le \tau \le 1), \quad p = \xi_0 \rho \tag{16}$$

and τ is an arbitrary constant.

Let us introduce as usual the mean anisotropy parameter A_m, the spatial volume V, the average scale factor and the Hubble parameter H for the metric (7)

$$V = a^3 = a_1 a_2 a_3 \tag{17}$$

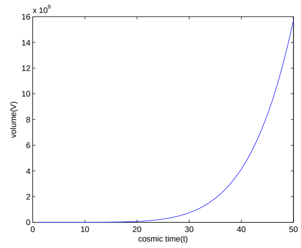

Fig. 1 The plot of volume V verses cosmic time t,with suitable constant value

$$H = \frac{1}{3}\frac{\dot{V}}{V} = \frac{1}{3}(\frac{\dot{a_1}}{a_1} + \frac{\dot{a_2}}{a_2} + \frac{\dot{a_3}}{a_3}), \tag{18}$$

$$\theta = 3H, \tag{19}$$

$$\sigma^2 = \frac{1}{2}\left[\left(\frac{\dot{a_1}}{a_1}\right)^2 + \left(\frac{\dot{a_2}}{a_2}\right)^2 + \left(\frac{\dot{a_3}}{a_3}\right)^2\right] - \frac{\theta^2}{6}, \tag{20}$$

$$A_m = \frac{1}{3}\Sigma_{i=1}^{3}\left(\frac{\Delta H_i}{H}\right)^2 \tag{21}$$

where $\Delta H_i = H_i - H$, i=1, 2, 3 and $H_1 = \frac{\dot{a_1}}{a_1}$, $H_2 = \frac{\dot{a_2}}{a_2}$ and $H_3 = \frac{\dot{a_3}}{a_3}$. An overdot denotes ordinary derivative with respect to t.

For the metric the field equations (6) together with (3) and (5), in (7) comoving coordinates lead to

$$\frac{\dot{a_1}\dot{a_2}}{a_1a_2} + \frac{\dot{a_2}\dot{a_3}}{a_2a_3} + \frac{\dot{a_3}\dot{a_1}}{a_3a_1} - \frac{m^2+m+1}{a_1^2} = \frac{1}{2\mu}\left[(5\bar{p}\mu + 16\pi\lambda) + (16\pi + 3\mu)\rho\right], \tag{22}$$

$$\frac{\ddot{a_2}}{a_2} + \frac{\ddot{a_3}}{a_3} + \frac{\dot{a_2}\dot{a_3}}{a_2a_3} - \frac{m}{a_1^2} = \frac{1}{2\mu}\left[\bar{p}(5\mu - 16\pi) + (\rho\mu + 8\pi\lambda)\right], \tag{23}$$

$$\frac{\ddot{a_1}}{a_1} + \frac{\ddot{a_3}}{a_3} + \frac{\dot{a_1}\dot{a_3}}{a_2a_3} - \frac{m^2}{a_1} = \frac{1}{2\mu}\left[\bar{p}(5\mu - 16\pi) + (\rho\mu + 8\pi\lambda)\right], \tag{24}$$

$$\frac{\ddot{a_1}}{a_1} + \frac{\ddot{a_2}}{a_2} + \frac{\dot{a_1}\dot{a_2}}{a_1a_2} - \frac{1}{a^2} = \frac{1}{2\mu}\left[\bar{p}(5\mu - 16\pi) + (\rho\mu + 8\pi\lambda)\right], \tag{25}$$

$$(m+1)\frac{\dot{a_1}}{a_1} - \frac{\dot{a_2}}{a_2} - m\frac{\dot{a_3}}{a_3} = 0. \tag{26}$$

3. Solution of Field Equations

For any physically relevant model, the Hubble parameter and deceleration parameter are the most important observational quantities in cosmology. Berman [58], and Berman and Gomide [59] have proposed a law of variation for Hubble parameter in FRW that yields a constant value of deceleration parameter and power-law and exponential forms of the average scale factor.

Recently, Akarsu and Dereli [60] proposed a special form for the deceleration parameter which is linear in time with a negative slope.This law covers the law of Berman used for obtaining exact of dark energy, to account for the current acceleration of the universe . According to this law only the spatially closed and flat universes with cosmological fluid exhibiting quintom like behavior are allowed and the universe ends with a big-rip.This new law gives the opportunity to generalize many of these dark energy models having better consistency with the cosmological observations .In recent papers, Adhav (2012), Sharif and Zubair (2012) and Chaubey and Shukla (2013) have generalized this assumption in anisotropic model. The deceleration parameter q is defined by

$$q = -\frac{a\ddot{a}}{\dot{a}^2} \tag{27}$$

related by the relation

$$q = -1 + \frac{d}{dH}(\frac{1}{H}). \tag{28}$$

This equation can be integrated to give the average scale factor

$$a(t) = e^{\delta} \exp \int \frac{dt}{\int (1+q)dt + \gamma} \tag{29}$$

where γ and δ are arbitrary constants of (29). We observe that q=constant is an easy choice which provides a(t) as an explicit function of t, discussed by Berman (1999). When q is taken to vary with time , an explicit determination of a(t) leads to a possible choice of q as Abdussattar and Prajapati [61] have presented a class of non-singular bouncing FRW models by constraining the deceleration parameter in the presence of an interacting dark energy represented by a time-varying cosmological constant.

$$q = -\frac{\alpha}{t^2} + (\beta - 1) \tag{30}$$

Here $\alpha > 0$ is a parameter having the dimension of square of time and $\beta > 1$ is a dimensionless constant. Obviously, the different values of α and β will give rise to different models. Equation (29) can be integrated to give the time variation of the scale factor as

$$a(t) = e^{\delta} \exp \left[\frac{1}{\beta} \int \left(\frac{tdt}{t^2 + t\frac{\gamma}{\beta} + \frac{\alpha}{\beta}} \right) \right] \tag{31}$$

The integral appearing in (31) can not be evaluated for arbitrary values of the constants. Setting $\gamma = 0$ in (31) and integrating we obtain the average scale factor a(t) as

$$a(t) = e^{\delta}(t^2 + \frac{\alpha}{\beta})^{\frac{1}{2\beta}}. \tag{32}$$

If we take $\alpha = 0$ and $e^{\delta}{=}D^{\frac{1}{\beta}}$ in (32), we obtain

$$a(t) = (Dt)^{\frac{1}{\beta}} \tag{33}$$

which corresponds to a with a constant deceleration parameter q=$\beta - 1$ throughout the evolution (Berman 1991).

It is difficult to solve equations $(22)-(26)$ for six unknown $a_1, a_2, a_3, \rho, \lambda$ and $\bar{p}$ in the exact form. In order to solve the system completely we assume that $a_3 = V^b$, where b is any constant number. From equations (17), (26), and (32), we obtain the expressions for the scale factors:

Model I

$$a_1(t) = (t^2 + \frac{\alpha}{\beta})^{\frac{3+3mb-3b}{2\beta(m+2)}}, \tag{34}$$

$$a_2(t) = (t^2 + \frac{\alpha}{\beta})^{\frac{3+3m-3b-6mb}{2\beta(m+2)}}, \tag{35}$$

$$a_3(t) = (t^2 + \frac{\alpha}{\beta})^{\frac{3b}{2\beta}}. \tag{36}$$

provided $m \neq -2$

For the model presented by the scale factors $(34)-(36)$, the kinematical parameters have values as given below:

$$H_1 = \frac{3+3mb-3b}{\beta(m+2)} \frac{t}{(t^2+\frac{\alpha}{\beta})}, \tag{37}$$

$$H_2 = \frac{3+3m-3b-6mb}{\beta(m+2)} \frac{t}{(t^2+\frac{\alpha}{\beta})}, \tag{38}$$

$$H_3 = \frac{3b}{\beta} \frac{t}{(t^2+\frac{\alpha}{\beta})}, \tag{39}$$

$$H = \frac{t}{\beta(t^2+\frac{\alpha}{\beta})}. \tag{40}$$

The expansion scalar, shear scalar and mean anisotropic parameter are obtained as

$$\theta = 3H = \frac{3t}{\beta(t^2+\frac{\alpha}{\beta})}, \tag{41}$$

$$\sigma^2 = A_1 \frac{t^2}{(t^2+\frac{\alpha}{\beta})^2}, \tag{42}$$

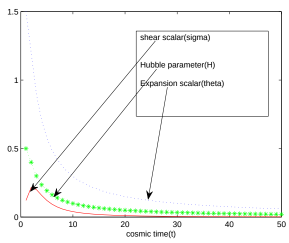

Fig. 2 The plot of Hubble parameter H, scalar expansion θ and shear scale σ verses cosmic time t

$$Am = \frac{2}{3(m+2)^2}[(3+3mb-3b)(6-3mb-6b+3m)+ \\ (3+3m-3b-6mb)(3+3m+3m-3mb)] + \frac{2}{3(m+2)^2}\left[(3b(m+2))(6mb+3b+3)\right] \tag{43}$$

where

$$A_1 = \frac{(3+3mb-3b)^2 + (3+3m-3b-6mb)^2 + 9(m+2)^2(b^2-2)}{2\beta^2(m+2)^2}$$

By using (22) and (23), we obtain bulk viscous pressure $\bar{p}$, string tension density λ and energy density ρ as r$\neq \pm 1$

$$\bar{p} = \left[\frac{2\mu t^2}{\beta^2(m+2)^2(t^2+\frac{\alpha}{\beta})^2[256\pi^2(1+r)+(-25-10\mu-17\mu\pi)\mu r]}\right][(8\pi+r\mu\lambda)M_1 \\ +(16\pi(1+r)+3\mu r)M_2 - \frac{(16\pi(1+r)+3\mu r)\alpha(\beta+3b)(m+2)^2}{t^2}] \\ -2\mu\left[\frac{(m^2+m+1)(8\pi+r\lambda\mu)-(16\pi(1+r)+3\mu r)}{(t^2+\frac{\alpha}{\beta})^{\frac{3+3mb-3b}{\beta(m+2)}}[256\pi^2(1+r)+(-25-10\mu-17\mu\pi)\mu r]}\right] \tag{44}$$

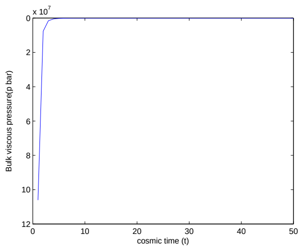

Fig. 3 The plot of bulk viscous pressure $\bar{p}$ verses cosmic time t

$$\lambda = \left(\frac{2\mu}{8\pi+r\mu}\right)\frac{t^2}{(t^2+\frac{\alpha}{\beta})^2\beta^2(m+2)^2}[M_3 - \frac{(5\mu-16\pi)}{(256\pi^2+(-25-10\mu-17\pi\mu)\mu r)}[(8\pi+r\mu)M_1+ \\ (16\pi(1+r)+3\mu r)M_2 - \frac{(16\pi(1+r)+3\mu r)\alpha(\beta+3b)(m+2)^2}{t^2} + (m+2)(\alpha+3b) \\ (3+3m-3b-6mb)+3b\alpha(m+2)]] - \left(\frac{2\mu}{8\pi+r\mu}\right)\left[\frac{m}{(t^2+\frac{\alpha}{\beta})^{\frac{3+3mb-3b}{\beta(m+2)}}}\right] \\ -\left(\frac{2\mu}{8\pi+r\mu}\right)\left[\frac{(m^2+m+1)(8\pi+r\lambda\mu)-(16\pi(1+r)+3\mu r)}{(t^2+\frac{\alpha}{\beta})^{\frac{3+3mb-3b}{\beta(m+2)}}[256\pi^2(1+r)+(-25-10\mu-17\mu\pi)\mu r]}\right] \tag{45}$$

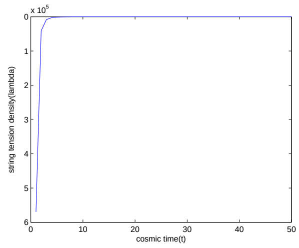

Fig. 4 The plot of string tension density λ verses cosmic time t

Using equation (12), we obtain

$$\rho = \left(\frac{2\mu r}{8\pi + r\mu}\right)\frac{t^2}{(t^2+\frac{\alpha}{\beta})^2\beta^2(m+2)^2}[M_3 - \frac{(5\mu - 16\pi)}{(256\pi^2 + (-25 - 10\mu - 17\pi\mu)\mu r)}[(8\pi + r\mu)M_1 + (16\pi(1+r) + 3\mu r)M_2 - \frac{(16\pi(1+r)+3\mu r)\alpha(\beta+3b)(m+2)^2}{t^2} + (m+2)(\alpha+3b)(3+3m-3b-6mb) + 3b\alpha(m+2)]] - \left(\frac{2\mu r}{8\pi + r\mu}\right)\left[\frac{m}{(t^2+\frac{\alpha}{\beta})^{\frac{3+3mb-3b}{\beta(m+2)}}}\right] + \left(\frac{2\mu r}{8\pi + r\mu}\right)\left[\frac{(m^2+m+1)(8\pi + r\lambda\mu) - (16\pi(1+r)+3\mu r)}{(t^2+\frac{\alpha}{\beta})^{\frac{3+3mb-3b}{\beta(m+2)}}[256\pi^2(1+r) + (-25 - 10\mu - 17\mu\pi)\mu r]}\right]. \tag{46}$$

In order to solve Einstein's field equations in the presence of cosmic strings we resort to a tractable assumption concerning a relation between the rest energy density and tension density of the system of strings. Using barotropic equation of state parameter as discussed in detailed (13), (14) and (15),the energy density of the particles attached to the string and bulk viscosity is :

$$\rho_p = (r-1)\left(\frac{2\mu}{8\pi + r\mu}\right)\frac{t^2}{(t^2+\frac{\alpha}{\beta})^2\beta^2(m+2)^2}[M_3 - \frac{(5\mu - 16\pi)}{(256\pi^2 + (-25 - 10\mu - 17\pi\mu)\mu r)}[(8\pi + r\mu)M_1 + (16\pi(1+r) + 3\mu r)M_2 - \frac{(16\pi(1+r)+3\mu r)\alpha(\beta+3b)(m+2)^2}{t^2} + (m+2)(\alpha+3b)(3+3m-3b-6mb) + 3b\alpha(m+2)]] - \left(\frac{2\mu(r-1)}{8\pi + r\mu}\right)\left[\frac{m}{(t^2+\frac{\alpha}{\beta})^{\frac{3+3mb-3b}{\beta(m+2)}}}\right] + \left(\frac{2\mu(r-1)}{8\pi + r\mu}\right)\left[\frac{(m^2+m+1)(8\pi + r\lambda\mu) - (16\pi(1+r)+3\mu r)}{(t^2+\frac{\alpha}{\beta})^{\frac{3+3mb-3b}{\beta(m+2)}}[256\pi^2(1+r) + (-25 - 10\mu - 17\mu\pi)\mu r]}\right] \tag{47}$$

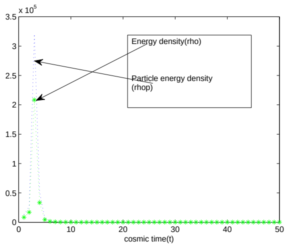

Fig. 5 The plot of energy density ρ and particle energy density ρ_p verses cosmic time t,with

$$\zeta = \left(\frac{2\mu}{8\pi + r\mu}\right)\frac{t}{3(t^2+\frac{\alpha}{\beta})\beta(m+2)^2}[M_3 - \frac{(5\mu - 16\pi)}{(256\pi^2 + (-25 - 10\mu - 17\pi\mu)\mu r)}[(8\pi + r\mu)M_1 +$$
$$(16\pi(1+r) + 3\mu r)M_2 - \frac{(16\pi(1+r) + 3\mu r)\alpha(\beta + 3b)(m+2)^2}{t} + (m+2)(\alpha + 3b)$$
$$(3 + 3m - 3b - 6mb) + 3b\alpha(m+2)]] - \left(\frac{2\mu\beta(t^2 + \frac{\alpha}{\beta})}{(8\pi + r\mu)3t}\right)\left[\frac{m}{(t^2+\frac{\alpha}{\beta})^{\frac{3+3mb-3b}{\beta(m+2)}}}\right]$$
$$+ \left(\frac{2\mu(t^2+\frac{\alpha}{\beta})}{(8\pi + r\mu)3t}\right)\left[\frac{(m^2+m+1)(8\pi + r\lambda\mu) - (16\pi(1+r) + 3\mu r)}{(t^2+\frac{\alpha}{\beta})^{\frac{3+3mb-3b}{\beta(m+2)}}[256\pi^2(1+r) + (-25 - 10\mu - 17\mu\pi)\mu r]}\right]. \tag{48}$$

The scalar curvature R is given by

$$R = 2\left[\frac{\ddot{a}_1}{a_1} + \frac{\ddot{a}_2}{a_2} + \frac{\ddot{a}_3}{a_3} + \frac{\dot{a}_1\dot{a}_2}{a_1 a_2} + \frac{\dot{a}_2\dot{a}_3}{a_2 a_3} + \frac{\dot{a}_1\dot{a}_3}{a_1 a_3} - \frac{(m^2+m+1)}{a_1^2}\right] \tag{49}$$

$$= \frac{2t^2 M_4}{\beta^2(t^2+\frac{\alpha}{\beta})^2(m+2)^2} + 2\left[\frac{(3+3mb-3b)\alpha}{\beta(m+2)} + \frac{\alpha(3+3m-3b-6mb)}{\beta(m+2)^2} + \frac{3b\beta}{\beta^2}\right]\frac{1}{(t^2+\frac{\alpha}{\beta})^2} - \frac{2(m^2+m+1)}{(t^2+\frac{\alpha}{\beta})^{\frac{3+3mb-3b}{\beta(m+2)}}} \tag{50}$$

where

$$M_1 = (3+3mb-3b)(3+3m-3b-6mb) + 3b(m+2)(b+3m-6b-3mb),$$

$$M_2 = (3+3m-3b-6mb)\beta(m+2) - (3+3m-3b-6mb)^2 + 3b\beta(m+2)^2 - 9b^2(m+2)^2 - 3b(3+3m-3b-6mb)(m+2),$$

$$M_3 = \beta(m+2)(3+3m-3b-6mb) + (3+3m-3b-6mb)^2 - 3b\beta(m+2) + 9b^2(m+2),$$

$$M_4 = (3+3mb) - 3b)^2 - (3+3mb-3b)\beta(m+2) - \beta(m+2)(3+3m-3b-6mb) + (3+3m-3b-6mb)^2 - 3b\beta + 9b^2 + (3+3mb-3b)(3+3m-3b-6mb) + 3b(m+2)(3+3m-3b-6mb) + 3b(m+2)(3+3mb-3b).$$

The expressions for the Hubble's parameters, scalar expansion θ, magnitude of shear scalar σ, are decreasing function of time at t=0, it become infinite and t$\to \infty$ these are zero. The spatial volume is increasing function of time. However anisotropic parameter is constant. From (46), (47), (48)and (49) it is noted that energy density ρ, particle energy density ρ_p , bulk viscosity coefficient ζ and scalar curvature are decreasing function of time and tend to zero large time t. The present model has no singularity at t=0. These behaviors of ρ and ρ_p are clearly depicted in fig. (5) as a representative case with appropriate choice of constants of integration and other physical parameters using reasonably well-known situations

We observe from (43) and (44) bulk viscous pressure $\bar{p}$ and the string tension density λ are increasing function of time and are always negative which approach zero at a large time . (Letelier 1983) pointed out that λ may be positive or negative. When $\lambda < 0$, the string phase of the universe disappears, i.e. we have an anisotropic fluid of particles. The behavior of λ is shown in fig.(4)

4. Model 2

Making use of $a_3 = V^b$ in (26) and (33), we obtain

$$a_1 = (Dt)^{\frac{6b+3(m+1)(1-b)}{\beta(m+1)(m+2)}}, \tag{51}$$

$$a_2 = (Dt)^{\frac{3(m+1)-3bm-3b(m+1)}{\beta(m+2)}}, \tag{52}$$

$$a_3 = (Dt)^{\frac{3b}{\beta}}. \tag{53}$$

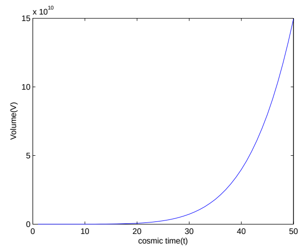

Fig. 6 The plot of volume V verses cosmic time t,with suitable constant value

with condition $m \neq -1, -2$

For the model given by (51),(52) and (53), the physical and kinematical parameters are given by

$$H = \frac{1}{3(m+1)(m+2)\beta t}[6b + 3b(m+1)(1-b) + (3(m+1) \\ -3bm - 3b(m+1))(m+1) + 3b(m+1)(m+2)], \tag{54}$$

$$\theta = \frac{1}{(m+1)(m+2)\beta t}[6b + 3b(m+1)(1-b) + (3(m+1) \\ -3bm - 3b(m+1))(m+1) + 3b(m+1)(m+2)], \tag{55}$$

$$\sigma^2 = \frac{A_2}{2\beta^2(m+1)^2(m+2)^2t^2}. \tag{56}$$

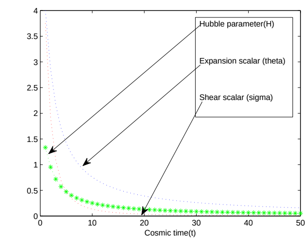

Fig. 7 The plot of Hubble parameter H, expansion scalar θ and shear scalar σ verses cosmic time t

$$A_m = \frac{3}{[6b+3(m+1)(1-b)+(3(m+1)-3bm-3b(m+1))(m+1)+3b(m+1)(m+2)]^2}$$
$$[(6b+3(m+1)(1-b))^2+(3(m+1)-3bm-3b(m+1))^2(m+1)^2+9b^2(m+1)^2(m+2)^2$$
$$-\frac{1}{3}[6b+3(m+1)(1-b)+(3(m+1)-3bm-3b(m+1))(m+1)+3b(m+1)(m+2)]^2] \tag{57}$$

where

$$A_2 = (6b+3(m+1)(1-b))^2+(3(m+1)-3bm-3b(m+1)^2+9b^2(m+1)^2(m+2)^2)$$
$$-2[(6b+3(m+1)(1-b)+(m+1)(3(m+1)-3bm-3b(m+1))+3b(m+1)(m+2))^2],$$

$$\bar{p} = \frac{2\mu}{(256\pi^2(1+r)+(-25-10\mu-17\pi\mu)\mu r)}[\left[\frac{1}{t^2\beta^2(m+1)^2(m+2)^2}\right][N_1(8\pi+r\mu)$$
$$-N_2(16\pi(1+r)+3\mu r)] - \frac{1}{(Dt)^{\frac{2(6b+3(m+1)(1-b))}{\beta(m+1)}}}[(16\pi(1+r)$$
$$+3\mu r)m+(m^2+m+1)(8\pi+r\mu)]] \tag{58}$$

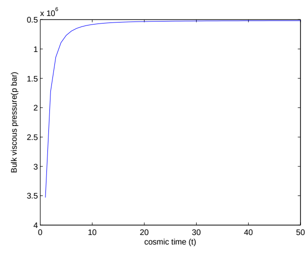

Fig. 8 The plot of bulk viscous pressure $\bar{p}$ verses cosmic time t,with suitable constant value

$$\begin{aligned}\lambda = \frac{2\mu}{(16\pi(1+r)+3\mu r)}[\frac{1}{\beta^2 t^2(m+2)^2(m+1)^2}[N_1 - \frac{1}{256\pi^2(1+r)+(-25-10\mu-17\mu\pi)\mu r} \\ ((8\pi+r\mu)N_1-(16\pi(1+r)+3\mu r)N_2)] - \frac{1}{(Dt)^{\frac{2(6b+3(m+1)(1-b))}{\beta(m+1)}}} \\ (m^2+m+1) - \frac{(16\pi(1+r)+3\mu r)m+(m^2+m+1)(8\pi+r\mu)}{256\pi^2(1+r)+(-25-10\mu-17\mu\pi)\mu r}],\end{aligned} \tag{59}$$

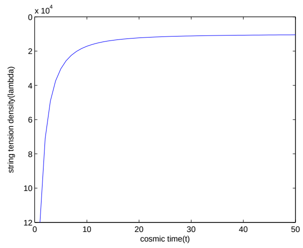

Fig. 9 The plot of string tension density λ verses cosmic time t

$$\begin{aligned}\rho = \frac{2\mu r}{(16\pi(1+r)+3\mu r)}[\frac{1}{\beta^2 t^2(m+2)^2(m+1)^2}[N_1 - \frac{1}{256\pi^2(1+r)+(-25-10\mu-17\mu\pi)\mu r} \\ ((8\pi+r\mu)N_1-(16\pi(1+r)+3\mu r)N_2)] - \frac{1}{(Dt)^{\frac{2(6b+3(m+1)(1-b))}{\beta(m+1)}}} \\ (m^2+m+1) - \frac{(16\pi(1+r)+3\mu r)m+(m^2+m+1)(8\pi+r\mu)}{256\pi^2(1+r)+(-25-10\mu-17\mu\pi)\mu r}],\end{aligned} \tag{60}$$

$$\begin{aligned}\rho_p = \frac{2(r-1)\mu}{(16\pi(1+r)+3\mu r)}[\frac{1}{\beta^2 t^2(m+2)^2(m+1)^2}[N_1 - \frac{1}{256\pi^2(1+r)+(-25-10\mu-17\mu\pi)\mu r} \\ ((8\pi+r\mu)N_1-(16\pi(1+r)+3\mu r)N_2)] - \frac{1}{(Dt)^{\frac{2(6b+3(m+1)(1-b))}{\beta(m+1)}}} \\ (m^2+m+1) - \frac{(16\pi(1+r)+3\mu r)m+(m^2+m+1)(8\pi+r\mu)}{256\pi^2(1+r)+(-25-10\mu-17\mu\pi)\mu r}],\end{aligned} \tag{61}$$

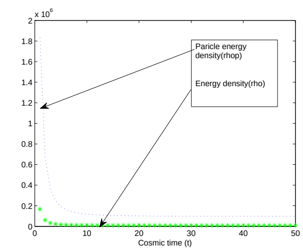

Fig. 10 The plot of energy density ρ and particle energy density ρ_p verses cosmic time t

$$\zeta = \left(\frac{2\mu r}{\beta t(m+1)(m+2)[6b+3(m+1)(1-b)+(3(m+1)-3bm-3b(m+1))(m+1)]} \right) *$$
$$[\left(\frac{r\xi_0}{(16\pi(1+r)+3\mu r)} - 1 \right) [N_1 - \frac{1}{256\pi^2(1+r)+(-25-10\mu-17\mu\pi)\mu r}$$
$$(8\pi + r\mu)N_1 - (16\pi(1+r)+3\mu r)N_2]] + \frac{1}{(Dt)^{\frac{2(6b+3(m+1)(1-b))}{\beta(m+1)}}} *$$
$$\left[(16\pi(1+r)+3r\mu)m + (m^2+m+1)(8\pi + r\mu)\right] *$$
$$\left[\frac{1-2r\mu(m+1)(m+2)}{(6b+3(m+1)(1-b)+(3(m+1)-3bm-3b(m+1))(m+1))} \right]$$
$$\left[\frac{2r\mu\xi_0(m^2+m+1)\beta t(m+1)(m+2)}{(Dt)^{\frac{2(6b+3(m+1)(1-b))}{\beta(m+1)}}(16\pi(1+r)+3\mu r)} \right] *$$
$$\left[\frac{1}{(6b+3(m+1)(1-b)+(3(m+1)-3bm-3b(m+1))(m+1)} \right], \tag{62}$$

$$R = \frac{2}{\beta^2(m+1)^2(m+2)^2t^2}[(6b+3(m+1)(1-b))(6b+3(m+1)(1-3b)$$
$$-\beta(m+1)(m+2)) + (3(m+1)-3bm-3b(m+1))(m+1)[(3(m+1)$$
$$-3mb-3b)(m+1)(m+2) - \beta(m+2)] + 3b(m+1)^2(m+2)^2$$
$$(3b-\beta) + (3(m+1)-3bm.3b(m+1))(6b+3(m+1)(1-b))(6b+3(m+1)$$
$$(1-b))(m+1)(m+2) + (3(m+1)-3bm-3b(m+1))3b(m+2)(m+1)^2]$$
$$\left[-2\frac{m^2+m+1}{(Dt)^{\frac{2(6b+3(m+1)(1-b))}{\beta(m+1)}}} \right] \tag{63}$$

where

$$N_1 = (m+1)[(3(m+1)-3bm-3b(m+1))(6b+3(m+1)(1-3b)+3b(m+1))$$
$$+3b(6b+3(m+1)(1-3b)+3b(m+1))],$$

$$N_2 = (6b+3(m+1)(1-3b)+3b(m+1))^2 - (6b+3(m+1)(1-3b)+3b(m+1))\beta(m+1)$$
$$(m+2) + (3(m+1)-3bm-3b(m+1))^2(m+1)^2(m+2)^2 - [(3(m+1)-3bm-3b(m+1))$$
$$\beta(m+2)(m+1)^2 + 9b^2(m+2)^2(m+1)^2 - 3b\beta(m+1)^2(m+2)^2.$$

The spatial volume of the model has the expression V=$(Dt)^{\frac{3}{\beta}}$ which is zero at t=0. At this epoch all the physical and kinematical parameters are infinite. Thus, the model starts expanding from a big-bang singularity at t=0. For $0 < t < \infty$. the physical and kinematical parameters are well definite and are decreasing function of time which ultimately tend to zero as $t \to \infty$. The behaviors of the physical and kinematical parameters are shown graphically. The anisotropy parameter is constant for all t.

Conclusion

In this paper we have studied spatially homogeneous and anisotropic cosmological models with bulk viscosity and cosmic strings in f(R,T) gravity theory. We have obtained two types of cosmological solutions of the field equations by setting the average scale factor and one of the scale factors equal to some power of the spatial volume of the model. One class of models is accelerated expanding universe having no finite singularity whereas the other class of models is also accelerating having finite singularity at the initial time t=0. In both types of models the physical and kinematical parameters are decreasing functions of time and tend to zero for large time. The anisotropy in the models is maintained throughout the passage of time. The bulk viscosity contributes negative pressure leading to an repulsive gravity which overcomes the attractive gravity and impetus for rapid expansion of the universe . We have particular classes of Bianchi models from the general class of Bianchi models considered in this paper, for different values of m as follows: Bianchi type-III corresponds to m=0, Bianchi type V corresponds to m=1, Bianchi type VI_0 corresponds to m=-1, and all other values of m give Bianchi type-VI_h.

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Physics of Currents and Potentials I. Classical Electrodynamics with Non-Point Charge

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Abstract: The formulation of classical electrodynamics, in which point charges are replaced by distributed current field, has been given. It has been demonstrated that 4-vector of current inside non-point particle is space-like and must be considered as a primary value which is not reduced to a charge carrier motion. A closed system of equations, describing simultaneous evolution of current and electro-magnetic field, has been introduced. The formula that makes calculation of mass ratio of different particles possible has been derived. There has been demonstrated the necessity to count space-time curvature inside non-point particles. The possibility of approximate preservation of the conception of point particle with spin moment with quasi-stationary description of the particle's movement in a weak electro-magnetic field has been shown. The solution of one of the particular problems of the constructed theory allows to suggest the existence of a new particle - heavy photon. One of the possible classical neutrino models, Maxwell neutrino, has been described. The theory requires the introduction of a fundamental constant which has the dimension of length.

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1. Introduction

As far as it is known, classical electrodynamics is an inherently contradictory theory. The contradictions are rooted in the combination of conceptions of the continuous field described by the equations in partial derivatives (Maxwell's equations) and the conceptions of singularities of fields-point charges, which motion is described by ordinary differential equations (Lorentz' equation). A. Einstein once called this combination as "unnatural technique" [1]. One way for decision of this contradiction is the removal of the field,

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description of electro-magnetic processes in the language of a moving charge. The names of Schwarzschild and Tetrode, Willer and Feynman (e.g. [2], [3]) are connected with this approach. This way does not lead to quite satisfactory results. Its non-availability is already obvious when it comes to dimensionality: the theory, containing electric charge and mass of a particle, automatically contains characteristic length, which undoubtedly will conflict with point nature of a charge.

The alternative way, which dates back up to Abraham, Lorentz and Poincare (e.g. [4]), - the refusal from point particles - is a construction of "purely" field theory. This way has never been popular, since some supplementary forces of non-electromagnetic nature had to be introduced to explain the charge stability.

One more attempt to solve this problem, which, according to V. L. Ginzburg's ironic definition [5], relates to the number of "eternal questions" of physics, is represented below [2,3]. We are dealing with the field theory of non-point distribution of matter. The difference from the predecessors' works consists of acceptance of the following heuristic point of view: electro-magnetic 4– current is a primary, fundamental quantity, which is not connected with the concepts of particles motion; energy-momentum tensor of charged matter is determined only by electro-magnetic current - any "contributions" of non-electro-magnetic origin are not necessary.

Let us consider anti-symmetric tensor of electro-magnetic field $F^{\mu\nu}$ and 4-vector of current density J^{μ} as the only observable quantities. Tensor $F^{\mu\nu}$ satisfies Maxwells equations:

$$\partial_\nu F^{\mu\nu} = -\frac{4\pi}{c} J^\mu, \tag{1}$$

$$\varepsilon^{\lambda\mu\nu\xi}\partial_\mu F_{\nu\xi} = 0, \tag{2}$$

and the current vector satisfies the equation of continuity:

$$\partial_\mu J^\mu = 0, \tag{3}$$

which is a condition for solvability of field equations (1), (2) (c is the velocity of light, $\varepsilon^{\lambda\mu\nu\xi}$ is a Levi-Civita symbol; the table of symbols corresponds to the book [6]]).

It is convenient to use the energy conservation law of the system "electro-magnetic field plus currents field" as a starting point for introduction of field equations of current field J^μ :

$$\partial_\nu T^{\mu\nu} = \frac{1}{c} F^{\mu\nu} J_\nu, \tag{4}$$

where $T^{\mu\nu}$ is the energy-momentum tensor of currents, depending on J^ν.

2. Form of tensor of current energy-momentum

The functional connection between $T^{\mu\nu}$ and J^ν is uniquely fixed by the simplest heuristic requirements: locality, the lowest order of equations; the lowest grades of non-linear

[2] The words "literally eternal questions" are omitted in English version of Ginzburg's book, but they exist in original Russian version.

[3] Niels Bohr used other term: Galgenproblem (Galgen - gallows, gibbet).

terms in equations. Locality means that the components $T^{\mu\nu}$ in the given world point depend on the current components (and possibly their derivatives) in the same point. The requirement of the lowest possible order removes the current derivatives from the list of tensor $T^{\mu\nu}$ arguments. The simplest possible symmetric tensor of the second order, which can be constructed from J^{ν} , has the following form:

$$T^{\mu\nu} = A(J^{\mu}J^{\nu} + b\mathrm{g}^{\mu\nu}J_{\lambda}J^{\lambda}), \tag{5}$$

where A and b are constants; $\mathrm{g}^{\mu\nu}$ is Galilei's metric tensor.
Tensor of mode (5) leads to the lowest grades of nonlinear terms in equations (4).
Non-dimensional constant b is easy to determine in the following way. Convolution of the equations (4) with J^{μ} current must result in the identity under the anti-symmetry of $F^{\mu\nu}$. Consequently,

$$J_{\mu}\partial_{\nu}T^{\mu\nu} \equiv 0, \tag{6}$$

which, taking into account (3), gives the following:

$$(1 + 2b)J_{\mu}J^{\nu}\partial_{\nu}J^{\mu} \equiv 0,$$

from which follows:

$$b = -\frac{1}{2}. \tag{7}$$

The sign of constant A is determined by the condition of positiveness of power density of the current field:

$$T^{00} \geq 0,$$

which, taking into account (5) and (7), gives the condition

$$A(\rho^2c^2 + \mathbf{J}^2) \geq 0, \tag{8}$$

from which follows:

$$A > 0. \tag{9}$$

In formula (8) standard notation is used:

$$J^{\mu} = \{\rho c, \mathbf{J}\},$$

where ρ is a charge density, and $\mathbf{J}$ is a 3- vector of current.
Further we shall denote $A = a^2$. Comparison of (1) with (4) and (5) shows that quantity a has time dimension. It is convenient to represent constant a as $a = \dfrac{r_0}{c}$, where r_0 is unknown fundamental constant of length dimension, characterizing interaction of current and electromagnetic field.
It is known that heavy leptons do not demonstrate any tracks of spatial structure up to the distance of about 10^{-16} centimeters. Therefore, $r_0 \leq 10^{-16}$ centimeters. However, we cannot judge how "strong" the inequality symbol is in this expression. A genuine

"lengths desert" [4] lies between this length and Planckian length ($\approx 10^{-33}$ centimeters). The condition of positiveness of tensor of current energy-momentum trace [6]:

$$T^{\mu}_{\mu} \geq 0,$$

combined with (5), (7) and (9), results in the following:

$$J^{\mu} J_{\mu} \leq 0. \tag{10}$$

The relation means that 4-current J^{μ} is space-like and, consequently, 3-current $\mathbf{J}$ cannot be interpreted as something generated by charge ρ motion.

The condition (10), which heuristic meaning, probably, oversteps the limits of specific mode of tensor $T^{\mu\nu}$ (5), shows that current $\mathbf{J}$ density inside non-point charged particle, similarly to the density of charge ρ, must be considered as the primary, fundamental concept, not reduced to any other simple one. Perhaps, the condition (10) is a "rather mad idea" which did not still appear in physics of the XXth century, and which necessity Niels Bohr once spoke about.

P.A.M. Dirac was very close to the formulation of such idea. In his classical work, written in 1938 (published in 1939), he formulated the following statement: "It is possible for a signal to be transmitted faster than light through the interior of an electron. The finite size of the electron now reappears in a new sense, the interior of the electron being a region of failure, not of the field equations of electromagnetic theory, but of some of the elementary properties of space-time" [7]. Undoubtedly, in fact, it concerns not the refusal of general properties of space-time (the theory is Lorentz-invariant), but the refusal of general interpretations of current as motion: 4-current J^{μ} is primary, non-interpreted object of the theory, just like 4-potential A^{μ}; the particles are not the objects "at the entrance" of the theory, - they may appear "at the exit" as their own states of current-potential fields.

We have started the development of the theory, proceeding from discussion of the formula of an energy-momentum tensor, affording not to agree with the classical position of A. Einstein and L. Infeld: "All attempts to represent matter by an energy-momentum tensor are unsatisfactory" [8]. On the contrary, we suppose that persistent tendency to avoid solution of this totally classical problem is responsible for "ugliness and incompleteness"[5] of the quantum relativistic theory, requiring renormalization and regularization.

3. Equations of current field

Substitution of (5) into (4) results in the following equations (for further simplicity let us assume $r_0 = 1$, $c = 1$):

$$J^{\nu} \partial_{\nu} J_{\mu} - J^{\nu} \partial_{\mu} J_{\nu} = F_{\mu\nu} J^{\nu}. \tag{11}$$

[4] In this desert there is only one distinguished dimension - the length of the Grand Unification, $\approx 10^{-29} cm$.

[5] P. A. Diracs words.

The inequality (10) implicitly defines 4- zone Ω_J of this system of equations action (current tube). Outside this zone current J^μ is absent, the electromagnetic field satisfies Maxwell's homogeneous equations, and the relation (11) turns into an empty identity.
As follows from the equation of continuity, the current tube is either closed (the birth and annihilation of charges system, neutral on the whole), or goes to infinity along the time base. Branching and coalescence of current tubes are possible. For example, horseshoe-shaped tube fits annihilation (both ends go to $t = -\infty$) or production of a pair (both ends go to $t = +\infty$).
The following boundary conditions are fulfilled on the three-dimensional space-like boundary σ:

$$J^\nu n_\nu = 0, \tag{12}$$

where n_ν is the 4- vector of normal to σ, and

$$n_\nu [F^{\mu\nu}] = 0, \qquad \varepsilon^{\mu\nu\lambda\xi} n_\nu [F_{\lambda\xi}] = 0, \tag{13}$$

where $[F^{\mu\nu}]$ is an electromagnetic field tensor jump on the surface σ, which satisfies the equation

$$J^\nu J_\nu = 0. \tag{14}$$

This boundary σ, satisfying the condition (14), separating space-time area occupied by currents, from the area free from currents, is very important in the theory. We shall name it with a special term ***pomerium***, - it was the name of the ancient Roman holy wall, surrounding the city [9].

Conditions (12) and (13) follow from the continuity equation (3) and Maxwell's equations (1), (2).

Disappearance of normal to surface σ 4-current component (12) results in energy-momentum flux absence through boundary σ.
Three-dimensional form of the notation (11) has the following view:

$$\begin{aligned} &\mathbf{J}\cdot\frac{\partial \mathbf{J}}{\partial t} + \mathbf{J}\cdot\bigtriangledown\rho = \mathbf{J}\cdot\mathbf{E}, \\ &\rho\cdot\frac{\partial \mathbf{J}}{\partial t} + \rho\cdot\bigtriangledown\rho + \mathrm{rot}\mathbf{J}\times\mathbf{J} = \rho\cdot\mathbf{E} + \mathbf{J}\times\mathbf{H}. \end{aligned} \tag{15}$$

where $\mathbf{E}$, $\mathbf{H}$ are 3-vectors of magnetic and electrical field intensity.
Operating zone of the equations is determined by the condition:

$$\mathbf{J}^2 \geq \rho^2. \tag{16}$$

Boundary conditions (12), (13) in three-dimensional notation take on the following form:

$$\begin{aligned} &\cos\gamma = \nu_n, \qquad \mathbf{n}\cdot[\mathbf{E}] = 0, \qquad \mathbf{n}\cdot[\mathbf{H}] = 0, \\ &\mathbf{n}\times[\mathbf{H}] = \nu_n\cdot[\mathbf{E}], \qquad [\mathbf{E}]\times\mathbf{n} = \nu_n\cdot[\mathbf{H}], \end{aligned} \tag{17}$$

where γ is the angle between 3-current $\mathbf{J}$ and unit 3-vector $\mathbf{n}$ of the normal to two-dimensional boundary S of 3-range V_J, occupied with currents; ν_n is a normal component of the boundary 3-velocity; $[\mathbf{E}]$, $[\mathbf{H}]$ are discontinuities of vectors $\mathbf{E}$, $\mathbf{H}$ during the

boundary passing. As follows from (14), the two-dimensional boundary of the current tube (pomerium), itself, is determined by the equation:

$$\mathbf{J}^2 = \rho^2. \tag{18}$$

It is possible to specify some equivalent forms of the equations notation (11). For example, using 4-potential of electromagnetic field A^μ, (11) the current can be removed from (11):

$$\Box A^\nu(\Box F_{\mu\nu} - 4\pi F_{\mu\nu}) = 0,$$

where $\Box$ is D' Alemberts operator: $\Box = -\partial_\mu \partial^\mu$.
Using general solution of Maxwell's equations as retarded potential, it is possible to exclude tensor $F^{\mu\nu}$ from (11), which will give a non-linear integro-differential equation relative to current J^μ.
Equations (11) are not sufficient for determination of 4-current J^μ, as the system (11) contains only three independent equations under the identity (6).
Equations (11) suggest the simplest possible form of field equations for the current:

$$\partial_\nu J_\mu - \partial_\mu J_\nu = F_{\mu\nu}, \tag{19}$$

or, in three-dimensional notation:

$$\begin{aligned} &\frac{\partial \mathbf{J}}{\partial t} + \bigtriangledown \rho = \mathbf{E}, \\ &\text{rot}\mathbf{J} = -\mathbf{H}. \end{aligned} \tag{20}$$

Equations (20) are as fundamental, original and "not derived from anything" for the given theory, as Maxwell's equations are.
We have to note that P. A. Dirac's article, dated by 1931 [10], contains the equations (20). P. A. Dirac's equations differ from those of ours (20) by constant multiplier, containing Planck's constant, and which is more important, by 4-vector J^μ interpretation: Dirac has it as 4-gradient of non-integrable phase of the wave function. P. Dirac's revolutionary article [10] introduced a new object into physics – magnetic monopole, which still has not been discovered by experimenters for the following eight decades. Probably, P. Dirac's overall attention to a new physical object, as well as his concentration on the problems of relativistic quantum theory in the early thirties, prevented him from seeing the possibility of classical interpretation of vector J^μ as a density vector of 4-current inside non-point particle.
System (19) contains six equations relatively to four current components. However, on account of Maxwell's equations (2), not all of them are independent. Equations (19) in combination with (1), (2), (3) make a closed set of equations relative to ten required quantities of J^μ, $F^{\mu\nu}$. The easiest way receive evidence of it is with help of a three-dimensional notation. This notation consists of two parts: the part, determining derivatives with re-

spect to time –

$$\begin{aligned}
\frac{\partial \mathbf{J}}{\partial t} &= \mathbf{E} - \bigtriangledown \rho, \\
\frac{\partial \rho}{\partial t} &= -\mathrm{div}\mathbf{J}, \\
\frac{\partial \mathbf{H}}{\partial t} &= -\mathrm{rot}\mathbf{E}, \\
\frac{\partial \mathbf{E}}{\partial t} &= \mathrm{rot}\mathbf{H} - 4\pi\mathbf{J},
\end{aligned} \tag{21}$$

and the part which does not contain any time derivatives

$$\begin{aligned}
&\mathrm{div}\mathbf{E} = 4\pi\rho, \\
&\mathrm{div}\mathbf{H} = 0, \\
&\mathrm{rot}\mathbf{J} = -\mathbf{H}.
\end{aligned} \tag{22}$$

It is obvious, that equations (22) play the role of restrictions, imposed on the initial conditions, as from (21) follows that, in case if relations (22) are performed at the initial moment, they are performed at any t.

Equations (19) mean that 4-current J^μ is quite simply connected with 4-potential A^μ:

$$J^\mu + A^\mu + \partial^\mu \chi = 0, \tag{23}$$

where χ is arbitrary scalar function of coordinates and time.

Equality (23) is performed only within the current tube; outside it, beyond pomerium, potential A^μ is defined by Maxwell's homogeneous equations and boundary conditions (12), (13).

Field equations in the form (23) and Maxwell's equations can be resulted from variational principle with help of Lagrangian L of the following form:

$$L = -\frac{1}{2} J^\mu J_\mu - J^\mu A_\mu - \frac{1}{16\pi} F_{\mu\nu} F^{\mu\nu}. \tag{24}$$

While obtaining field equations from Lagrangian's density (24), it is necessary to take into account that the charge conservation law (3) imposes an additional condition, which has to be taken into consideration at minimization of action functional. This means that supplementary component must be added to Lagrangian's density (24):

$$L_{ad} = -\chi \partial_\mu J^\mu, \tag{25}$$

in which $\chi = \chi(x^\nu)$ is Lagrange multiplier that constitutes the function of space-time coordinates x^ν in zone Ω_J, occupied by current J^μ.

We shall transform the right part (25), extracting 4-divergence:

$$\chi \partial_\mu J^\mu = \partial_\mu (\chi J^\mu) - J_\mu \partial^\mu \chi.$$

The first component in this formula vanishes after integration in zone Ω_J and, consequently, the supplementary component in Lagrangian can be rewritten in the following form:

$$L_{ad} = -J_\mu \partial^\mu \chi. \tag{26}$$

In the total, the effective Lagrangian L' has the form of the sum:

$$L' = L + L_{ad} = -\frac{1}{2}J_\mu J^\mu - J_\mu \partial^\mu \chi - J^\mu A_\mu - \frac{1}{16\pi}F^{\mu\nu}F_{\mu\nu}. \tag{27}$$

The simplicity of Lagrangian (27) and of the appropriate field equations (19) or (23) looks rather suspicious. However, it is not easy to set some other field equations, which would result in the energy conservation law (11) as naturally as (19) does.
Action functional S, corresponding to Lagrangians density (27) has the form:

$$S = -\int_{\Omega_J}\left(\frac{1}{2}J^\nu J_\nu + J_\nu\left(\partial^\nu\chi + A^\nu\right)\right)\mathrm{d}\Omega - \frac{1}{16\pi}\int_\infty F^{\mu\nu}F_{\mu\nu}\mathrm{d}\Omega. \tag{28}$$

In formula (28) the first integral is taken from 4-domain of the current tube (current tubes), the second integral is taken from the whole 4-space, including the domain inside the current tube (current tubes).
By minimizing the functional (28) and removing tensor $F^{\mu\nu}$ from the problem, we can reduce the problems of electro-dynamics to making a dynamically conjugated pair (J^ν, A^ν); obeying to the following equations in zone Ω_J:

$$\begin{aligned} &J^\nu + A^\nu + \partial^\nu\chi = 0,\\ &\Box A^\nu + \partial^\nu l = -4\pi J^\nu,\\ &\partial_\nu J^\nu = 0, \end{aligned} \tag{29}$$

(under fulfillment of the condition $J^\nu J_\nu \le 0$).
Outside zone Ω_J:

$$\Box A^\nu + \partial^\nu l = 0. \tag{30}$$

In these equations l is a 4-divergence of 4-potential A^μ:

$$l = \partial_\mu A^\mu.$$

Equations (29) represent the system of nine equations for determination of nine unknown functions of J^μ, A^μ, χ.
However, these equations are not independent: charge conservation law follows from Maxwell's equations. Therefore, it enables us to select Lagrange multiplier arbitrarily. Assuming $\chi = 0$ (we shall name such a choice of χ a "Lorentz gauge"), we discover from the first and the last equations of the system (29), that

$$l = 0 \quad \text{in range} \quad \Omega_J.$$

Demanding continuity of 4-vector A^μ and its derivatives on 3-surface σ, which limits 4-zone Ω_J, we can spread this condition on the total 4- space.
Finally, setting the task of classical electro-dynamics is reduced to the solution of the following problem: inside zone Ω_J $(J^\nu J_\nu \le 0)$:

$$\begin{aligned} &J^\nu + A^\nu = 0,\\ &\Box A^\nu = -4\pi J^\nu,\\ &\partial_\nu J^\nu = 0, \end{aligned} \tag{31}$$

outside zone Ω_J:

$$\begin{aligned} &\Box A^{\nu} = 0, \\ &\partial_{\mu} A^{\mu} = 0. \end{aligned} \tag{32}$$

4. Evolutionary and stationary problems of electrodynamics

Two types of problems can be formulated for the system (31), (32).
1) Evolutionary problem (Cauchy problem). "Initial' conditions $J^{\mu} = J^{\mu}_{(0)}$, $A^{\mu} = A^{\mu}_{(0)}$ are specified on some infinite time-like 3-surface. We must find J^{μ}, A^{μ} in the total 4-space.
"Initial" conditions must be consistent with the inequality (10) and the first equation of the system (21).
Full 4-momentum and angular momentum tensor are conserved for the system "current+electromagnetic field" in the evolutionary problem. Separately for the current or for the field, momentum and angular momentum are not conserved, varying while passing from one infinite time-like hyper-surface Σ to another.
2) Stationary problem. The formulated theory assumes stationary solutions in which momentum and moment of current field (and, automatically-momentum and moment of electro-magnetic field) are preserved. Such solutions describe the structure of a single non-point charge.
Condition of stationarity can be represented in the following form:

$$\begin{aligned} &\int_{\Delta\Omega} F^{\mu\nu} J_{\nu} \mathrm{d}\Omega = 0, \\ &\int_{\Delta\Omega} J_{\lambda} \left(x^{\mu} F^{\nu\lambda} - x^{\nu} F^{\mu\lambda} \right) \mathrm{d}\Omega = 0, \end{aligned} \tag{33}$$

where $\Delta\Omega$ is an arbitrary infinitely small 4-zone spanned on the arbitrary infinite time-like 3-surface Σ .
Condition (33) can be rewritten as the vanishing surface integrals:

$$\begin{aligned} &\int_{\Sigma} F^{\mu\nu} J_{\nu} \mathrm{d}\Sigma = 0, \\ &\int_{\Sigma} J_{\lambda} \left(x^{\mu} F^{\nu\lambda} - x^{\nu} F^{\mu\lambda} \right) \mathrm{d}\Sigma = 0, \end{aligned} \tag{34}$$

where $\mathrm{d}\Sigma$ is a scalar invariant of the oriented element of surface $\mathrm{d}\Sigma_{\mu}$

$$\mathrm{d}\Sigma^2 = \mathrm{d}\Sigma^{\mu} \mathrm{d}\Sigma_{\mu}.$$

Practically, integrals (34) are calculated not over the whole surface Σ, but only in the zone of Σ and Ω_J intersection.
In particular, there can be selected such reference frame (charge rest frame), in which the boundary of 3-area V_J, occupied by points, is stationary, and the derivatives of all quantities in time come to nothing.
Full three-dimensional formula of the stationary problem in the rest frame, under (21),

(22), has the form:
In zone V_J -

$$\begin{aligned} &\Delta\rho = 4\pi\rho, \\ &\Delta\mathbf{J} = 4\pi\mathbf{J}, \\ &\mathrm{div}\mathbf{J} = 0, \\ &\phi + \rho = 0, \\ &\mathbf{J} + \mathbf{A} = 0, \end{aligned} \tag{35}$$

$$\rho^2 \leq \mathbf{J}^2; \tag{36}$$

outside zone V_J Maxwell's homogeneous equations are performed:

$$\begin{aligned} &\Delta\phi = 0, \\ &\Delta\mathbf{A} = 0, \\ &\mathrm{div}\mathbf{A} = 0. \end{aligned} \tag{37}$$

Pomerium, i.e. boundary S of zone V_J is implicitly defined by the condition

$$\mathbf{J}^2 = \rho^2. \tag{38}$$

Boundary conditions (17) in the stationary problem have the following form:

$$\begin{aligned} &\mathbf{J} \cdot \mathbf{n} = 0; \\ &\mathbf{E}, \mathbf{H} \quad \text{continuous on} \quad S, \\ &\text{i.e} \quad \mathbf{E}^+ = (\bigtriangledown\rho)^-, \quad \mathbf{H}^+ = -(\mathrm{rot}\mathbf{J})^-, \end{aligned} \tag{39}$$

where symbols $\pm$ refer to external (+) and internal side of pomerium S surface.

It is not difficult to check that conditions of stationarity (33), which can be represented in the form of integrals in 3-volume V_J, are compatible with condition $\frac{\partial}{\partial t}$ and the boundary conditions of the problem.

5. Normalization of stationary solutions

Stationary problem is reduced to solution of homogeneous system of equations with homogeneous conditions. Nontrivial solutions are made with nonzero normalization. A few ways of normalization can be used, for example:
1. Linear normalizations
1a. Normalization by charge e -

$$\int_{V_J} \rho \mathrm{d}V = e. \tag{40}$$

1b. Normalization by magnetic moment $\boldsymbol{\mu}$ value -

$$\frac{1}{2} \left| \int_{V_J} \mathbf{x} \times \mathbf{J} \mathrm{d}V \right| = \mid \boldsymbol{\mu} \mid . \tag{41}$$

2. Quadratic normalization
2a. Normalization by energy (rest mass) m -

$$\frac{1}{2}\int_{V_J}\left(\rho^2+\mathbf{J}^2\right)\mathrm{d}V+\frac{1}{8\pi}\int_{\infty}\left(\mathbf{E}^2+\mathbf{H}^2\right)\mathrm{d}V=m. \tag{42}$$

2b. Normalization by value of angular momentum $\boldsymbol{M}$ -

$$\left|\int_{V_J}\rho\left(\mathbf{x}\times\mathbf{J}\right)\mathrm{d}V+\int_{\infty}\mathbf{x}\times\mathbf{S}\mathrm{d}V\right|=\mid\mathbf{M}\mid. \tag{43}$$

($\mathbf{x}$ - three-dimensional radius vector, $\mathbf{S}$ - Pointing vector: $\mathbf{S}=\frac{1}{4\pi}\mathbf{E}\times\mathbf{H}$)[6].
These formulas imply integration by volume V_J, occupied by currents and integration by the whole three-dimensional space for quantities, characterizing electro-magnetic field; and in volume V_J:

$$\mathbf{E}=\bigtriangledown\rho,\quad \mathbf{H}=-\mathrm{rot}\mathbf{J}.$$

As the equation system (25) disintegrates into independent equations relative to ρ and $\mathbf{J}$ (connected by algebraic condition (36) and boundary conditions), generally speaking, it is necessary to use two normalization conditions (one linear and one quadratic).

We did not manage to prove any exact assertions as to existence or a number of stationary problem solutions. However, it should be noted that this is a classical problem of splicing solutions of two equation systems (e.g., (35) and (37)) on an unknown boundary. Problems of this kind can often be met in different fields of continuum mechanics and they often have nonunique solution.

Different solutions vary by configuration of zone V_J (boundary connectivity of this zone).

6. Non-existence of nonsingular solutions of the stationary problem

However, quadratic normalizations of condition (42) or (43) cannot be imposed on the solution at all. It is not difficult to make sure that these integrals diverge.
For example, let us examine integral $\mathbf{E}^2$, entering into (42), all over the three-dimensional space.
Using equality

$$\mathbf{E}=-\bigtriangledown\varphi,$$

where φ is a scalar potential ($A^\mu=\{\varphi;\mathbf{A}\}$), we can perform the following transformations of this integral, assuming that the integral converges and, correspondingly, Gauss theorem

[6] Formula (43) - if we do not pay attention to the divergence of the integrals in it - allows to give a clear physical interpretation of the concept of "spin" of elementary particle without the use of representations of any mechanical motion. However, the theory under development, being *classical*, contains no clues for spin quantization.

can be applied:

$$\frac{1}{8\pi}\int_{\infty}\mathbf{E}^2\mathrm{d}V=-\frac{1}{8\pi}\int_{\infty}\mathbf{E}\cdot\bigtriangledown\varphi\mathrm{d}V=-\frac{1}{8\pi}\int_{\infty}\mathrm{div}\left(\mathbf{E}\varphi\right)\mathrm{d}V+\frac{1}{8\pi}\int_{\infty}\varphi\mathrm{div}\mathbf{E}\mathrm{d}V.$$

According to Gauss theorem:

$$\int_{\infty}\mathrm{div}\left(\mathbf{E}\varphi\right)\mathrm{d}V=\oint_{\infty}\mathbf{E}\varphi\mathrm{d}\mathbf{S}.$$

where infinitely far surface integration is made in its right side.
Taking into account that in zone V_J

$$\mathrm{div}\mathbf{E}=4\pi\rho$$

and $\varphi=-\rho$ (in accordance with the first equation (31)), we finally come to the following:

$$\frac{1}{8\pi}\int_{\infty}\mathbf{E}^2\mathrm{d}V=-\frac{1}{2}\int_{V_J}\rho^2\mathrm{d}V. \tag{44}$$

In formula (44) a deliberately positive value is in the left side, and a deliberately negative one is in the right side. Therefore, appropriate integrals are divergent and such a "free' application of Gauss theorem is incorrect. Consequently, it is wrong to assume that there is a regular solution of the stationary problem with finite inherent mass of non-point particle.
Virtually, this fact must not be considered as a drawback of the *classical* theory. On the contrary, existence of regular solutions with finite inherent mass in this theory would cause confusion: why did not God use the opportunity to create classical world with zero Plancks constant? Nonexistence of regular stationary solutions with finite own mass means that God just did not have such an opportunity.
However, remaining within the conventional frames of unquantized theory, we must learn to elicit useful information from divergent normalization integrals.
Current field singularities, which inevitably exist in the stationary problem solution, are concentrated on some lines in zone V_J. These lines can be either closed or open; in the latter case they start and finish on surface σ_J, which limits the volume of current zone V_J. On these singularity lines each or some of the components of current density $\mathbf{J}$ and charge density ρ turn into infinity.
These "singularity lines", existing inside current zone V_J, in some way look like the objects, regarded in the string theory. However, the "string" is a "bare" singularity, surrounded only by created by it electromagnetic field. "Singularity lines", presented in the regarded theory, are surrounded not only by electromagnetic field, but also by current field, which has a smooth, regular nature outside the thin tube of arbitrarily minor radius, covering the singularity line.
Let us denote characteristic radius of such tube with symbol ε. The boundary of this tube will be denoted as σ_ε. Without limiting generality, we can assume that σ_ε is the current surface, i.e., current vector $\mathbf{J}$ on σ_ε is tangential everywhere. Integration all over the

three-dimensional space with the deduction of zone V_ε enclosed inside tube σ_ε boundary, will be denoted with symbol $\int_{\infty'} \mathrm{d}V$, and current zone V_ε integration with the deduction of zone V_ε will be denoted with symbol $\int_{V_J} \mathrm{d}V$.

Electrostatic energy of the stationary solution outside singular tube is represented with the following formula:

$$U_\varepsilon = \frac{1}{8\pi}\int_{\infty'} \mathbf{E}^2 \mathrm{d}V. \tag{45}$$

Integration zone in (45) does not contain any singularities, therefore Gauss theorem application is correct in it for integral transformation:

$$U_\varepsilon = -\frac{1}{8\pi}\int_{\infty'} \mathrm{div}\left(\mathbf{E}\varphi\right)\mathrm{d}V + \frac{1}{8\pi}\int_{\infty'} \varphi \mathrm{div}\mathbf{E}\mathrm{d}V \tag{46}$$

or

$$U_\varepsilon = -\frac{1}{8\pi}\oint_{\sigma_\varepsilon} \varphi\mathbf{E}\cdot\mathbf{n}\mathrm{d}\sigma - \frac{1}{2}\int_{V_J'} \rho^2 \mathrm{d}V. \tag{47}$$

The first integral in the right side (46) is transformed at passing to (47) with respect to Gauss theorem; integral contribution on infinitely remote surface is omitted as a vanishing one; what remains is (a singular at $\varepsilon \longrightarrow 0$) contribution on surface σ_ε, in which vector $\mathbf{n}$ is a unit vector of a normal to surface σ_ε, directed *from* the singular line *inside* towards current zone V_J'.

Taking into account that

$$\mathbf{E}\cdot\mathbf{n} = -\frac{\partial\varphi}{\partial n} = \frac{\partial\rho}{\partial n},$$

and, consequently,

$$\varphi\mathbf{E}\cdot\mathbf{n} = \varphi\frac{\partial\rho}{\partial n} = -\rho\frac{\partial\rho}{\partial n};$$

where symbol $\dfrac{\partial}{\partial n}$ means a derivative with respect to a normal to surface σ_ε, the derived relations can be represented in the following form, which makes it possible to calculate the "electric part' of charge inherent mass

$$E_e' = \frac{1}{8\pi}\int_{\infty'} \mathbf{E}^2 \mathrm{d}V + \frac{1}{2}\int_{V_J'} \rho^2 \mathrm{d}V = -\frac{1}{2}\oint \frac{\partial\rho^2}{\partial n}\mathrm{d}\sigma. \tag{48}$$

With moving away from the singular line, ρ^2 decreases, on surface σ_ε derivative $\dfrac{\partial\rho^2}{\partial n} < 0$. Consequently, in relation (48) on the left and on the right from the right sign of equality there are positive values (aiming at $+\infty$ with $\varepsilon \longrightarrow 0$).

"Magnetic part" of inherent mass of particle ε_m' outside the limits of "singular tube" V_ε can be calculated in a similar way:

$$E_m' = \frac{1}{8\pi}\int_{\infty'} \mathbf{H}^2 \mathrm{d}V + \frac{1}{2}\int_{V_J'} \mathbf{J}^2 \mathrm{d}V = -\frac{1}{2}\oint_{\sigma_\varepsilon} \frac{\partial}{\partial n}\left(\mathbf{J}^2\right)\mathrm{d}\sigma. \tag{49}$$

So, to calculate inherent mass of particle m_ε outside the limits of "singularity tube" V_ε, it is sufficient to calculate surface integral over the surface of this tube σ_ε:

$$m_\varepsilon = -\oint_{\sigma_\varepsilon} \frac{\partial}{\partial n}\left(\frac{\rho^2 + \mathbf{J}^2}{2}\right)\mathrm{d}\sigma. \tag{50}$$

7. The possibility to calculate muon mass

Full own mass

$$m = \lim_{\varepsilon\to 0} m_\varepsilon,$$

is, naturally, infinite, however, we are able to calculate *ratio* of two particles masses, i.e. of two different stationary solutions by substituting, in the spirit of Cauchy-L'Hospital, ratio of non-existing limits with existing limit of ratio:

$$\frac{m_{II}}{m_I} = \lim_{\varepsilon\to 0} \frac{\oint_{\sigma_\varepsilon} \frac{\partial}{\partial n}\left(\frac{\rho^2 + \mathbf{J}^2}{2}\right) \mathrm{d}\sigma|_{II}}{\oint_{\sigma_\varepsilon} \frac{\partial}{\partial n}\left(\frac{\rho^2 + \mathbf{J}^2}{2}\right) \mathrm{d}\sigma|_{I}}. \tag{51}$$

In relation (51) by symbols I and II we denoted two different own solutions of the stationary problem which is different in current space topology and its boundary-pomerium (for example, sphere topology and torus topology) and, probably, in a number of singular lines, - with identical normalization by charge and magnetic moment and by identical vanishing velocity ε.

Formula (51) is the one that emerges in physics for the first time, it makes it possible to calculate ratio of heavy leptons masses (ratio of muon mass to electron mass; ratio of triton mass to electron mass)[7]. Moreover, the examined stationary problem formulation allows to answer the question "What is leptons?" (and in this very way to answer professor Isidor Isaak Rabi's famous question regarding muons: (see, for example, [12], chapter 1, "Introduction") "Who ordered that?" *Heavy leptons are own solutions of the above set stationary problem of classical electrodynamics of non-point particles, varying in pomerium configuration and, probably, in a number of singular lines inside pomerium.*

So, each heavy lepton is characterized by a couple of integral numbers-it is the connectivity of its boundary and a number of singularity lines (if stationary solutions, containing more than one singularity line, are possible). Within the frames of the made picture, there are no obvious grounds to assume that there exist only *three* heavy leptons (and even that their number is finite): along with τ- lepton or triton ($\tau\rho\iota\tau o\nu$ means "for the third time" from Greek) a more massive $\tau\varepsilon\tau\alpha\tau\rho o\nu$ ("for the fourth time") can exist, etc.

If the created theory can be "quantized", then during the process of quantization, visual idea about existence of certain distribution of currents in space will, undoubtedly, vanish – like during quantization of Keplerian problem, visual idea about ellipsoid path of electron in hydrogen atom vanishes – but there must remain quantum numbers, characterizing lepton, not having more visual interpretation in the view of connectivity of boundaries and number of singular lines.

It goes without saying that without technically proved theorem about spectrum of stationary problem solutions, it is impossible to insist either on the provided here correctness of leptons definition, or on the application of formula (51) to calculation of some

[7] "Qualis alio modo repereri non potest", - ("it is impossible to find out in other mode") so, a bit boastfully, Nikolaj Kopernik characterized the possibilities of his theory (quotation [11, p. 54]).

particles mass. In particular, if the stationary problem solution is unique, there exists only one massive lepton in electromagnetic (Maxwells) sector of physics; other leptons are not absolute Maxwells objects.

8. Behavior of solutions in the vicinity of singular lines

Let us give consideration to the local structure of non-point particle close to singular lines. Let us introduce local cylindrical co-ordinates r, λ, z, with axis z, tangential towards the singular line. Supposing that the solutions have local axial symmetry and, ignoring longitudinal derivative in z-co-ordinate compared to radial derivatives, equation system (35) can be approximately transformed to a local system of ordinary differential equations with a radial co-ordinate as the only argument:

$$\begin{cases} r^2u'' + ru' - 4\pi r^2 u = 0, \\ r^2v'' + rv' - (4\pi r^2 + 1)v = 0. \end{cases} \tag{52}$$

Here $u = \rho$ or $u = J_z$; $v = J_\lambda$, dash means a derivative by r. Radial current component vanishes in this approximate description. Equation (52) is Bessels modified equations. Writing out only the principal singular parts of the solution, dominating with $r \longrightarrow 0$, we shall represent the local solution (52) as the following:

$$\begin{aligned} J_\lambda &= P \cdot r^{-1}, \\ J_z &= Q \cdot \ln r, \\ \rho &= R \cdot \ln r. \end{aligned} \tag{53}$$

In relations (53) P, Q, R are slowly changing functions of curved co-ordinate l, directed along singular line. The view of these functions is determined by the complete solution of the stationary problem, and it cannot be determined from local analysis. In view of different character of the singularity of azimuthal current component J_λ and axial component , we should separately examine categories of solutions with function $P(l)$, which is different from identical null (P-solutions), and categories of solutions with $P \equiv 0$ (Q -solutions). For P-solutions, the condition of space-like current (36) is deliberately fulfilled in vicinity of the singular line. For Q-solutions (if there are any) from condition (36) follows that $Q^2 \geq R^2$.

Calculating by (53) the integrals, entering into expressions (50) and (51) for the own mass of non-point particle, we find out within the frames of the examined approximation that the value m_ε can be represented in the following form:

$$m_\varepsilon = 2\pi \left(\ln \frac{1}{\varepsilon} \cdot \oint \left(Q^2 + R^2\right) \mathrm{d}l + \frac{1}{\varepsilon^2} \oint P^2 \mathrm{d}l \right). \tag{54}$$

In formula (54) it is supposed that the curvilinear integral is taken along the singular line. It follows from (54) that it is possible to calculate the ratio of masses for two of P-solutions or for two of Q-solutions, but it is impossible to calculate the ratio of the

own masses for solutions of different classes – one P-solution, another one – Q-solution. For P-solutions, from (53) and (54) the approximate formula of ratio of two P-solutions masses, follows:

$$\frac{m_{II}}{m_I} = \frac{\oint P_{II}^2(l)\mathrm{d}l}{\oint P_I^2(l)\mathrm{d}l}. \tag{55}$$

In formula (55) P_I and P_{II} are - intensity of singularity of azimuthal current component J_λ, included into (53).

9. Inevitability of space-time curvature in current zone Ω_J

Existence of singular lines inside current zone Ω_J generates unlimited growth of energy-momentum tensor $T^{\mu\nu}$ (both for current field and for electromagnetic field). According to Einstein's equations of gravitation theory, this causes unlimited growth of the components of Ricci tensor $R^{\mu\nu}$ [6].

$$R^{\mu\nu} = k\left(T^{\mu\nu} - \frac{1}{2}\mathrm{g}^{\mu\nu}T\right) \tag{56}$$

($T = T^{\mu}{}_{\mu}$ is the trace of energy-momentum tensor), where k is a non-dimensional constant proportional to the constant of Newtonian attraction (universal gravitation) G.
In units of measurement which are used here ($r_0 = 1$, $c = 1$), the coordinates, metric tensor components, Christoffel symbols and the components of Ricci tensor $R^{\mu\nu}$ are non-dimensional quantities; the dimension of current J^ν and potential A^ν coincides with the dimension of electric charge. If currents and potentials are attributed to the quantity of charge of electron e, the current, the potential, the components of electro-magnetic field also become non-dimensional quantities. With regard to this, non-dimensional constant k, included into Einstein's equation (56), has the following form through dimensional constants of physics:

$$k = 8\pi\frac{Ge^2}{r_0^2c^4}. \tag{57}$$

The consequence of Ricci tensor components growth is the impossibility to use Minkowski's global coordinates. The equations, previously written out in these coordinates, assuming global Lorentz invariance, should be rewritten in optional curvilinear coordinates with a priori unknown metric tensor $\mathrm{g}^{\mu\nu}$, which components satisfy Einstein's differential equations (56).
Inside zone Ω_J ($J_\mu J^\mu \leq 0$)

$$\begin{aligned} &J^\mu + A^\mu = 0, \\ &\partial_\mu\left(\sqrt{-\mathrm{g}}J^\mu\right) = 0, \\ &\partial_\nu\left(\sqrt{-\mathrm{g}}\left(\partial^\mu A^\nu - \partial^\nu A^\mu\right)\right) = -4\pi\sqrt{-\mathrm{g}}J^\mu. \end{aligned}$$

Outside zone Ω_J ($J^\mu \equiv 0$)

$$\begin{aligned} &\partial_\nu\left(\sqrt{-\mathrm{g}}\left(\partial^\mu A^\nu - \partial^\nu A^\mu\right)\right) = 0, \\ &\partial_\mu\left(\sqrt{-\mathrm{g}}A^\mu\right) = 0. \end{aligned} \tag{58}$$

In equations (58) g is a determinant of metric tensor $g^{\mu\nu}$.
Energy-momentum tensor $T^{\mu\nu}$, appearing in the right sides of Einstein's equations (56), is the sum of energy-momentum tensor of current field $T^{\mu\nu}_{cur}$ and energy-momentum tensor of electro-magnetic field $T^{\mu\nu}_{f}$:

$$\begin{aligned} T^{\mu\nu}_{cur} &= J^\mu J^\nu - \frac{1}{2} g^{\mu\nu} J_\lambda J^\lambda, \\ T^{\mu\nu}_{f} &= \frac{1}{4\pi}\left(-F^{\mu\lambda}F^{\nu\xi}g_{\lambda\xi} + \frac{1}{4}F_{\lambda\xi}F^{\lambda\xi}g^{\mu\nu}\right). \end{aligned} \tag{59}$$

The coupling of electro-magnetic field tensor $F^{\mu\nu}$ components and 4-potential A^μ components in the curvilinear coordinates is the same as in Minkowski's coordinates [6]

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{60}$$

If we also take into account the known relations, connecting Ricci tensor $R^{\mu\nu}$ with Christoffel symbol $\Gamma^\mu_{\nu\lambda}$ and Christoffel symbols with metric tensor $g^{\mu\nu}$ components [6]:

$$\begin{aligned} R_{\mu\nu} &= \partial_\lambda \Gamma^\lambda_{\mu\nu} - \partial_\nu \Gamma^\lambda_{\mu\nu} + \Gamma^\lambda_{\mu\nu}\Gamma^\xi_{\lambda\xi} - \Gamma^\lambda_{\mu\xi}\Gamma^\xi_{\nu\lambda}, \\ \Gamma^\mu_{\nu\lambda} &= \frac{1}{2} g^{\mu\xi}\left(\partial_\lambda g_{\xi\nu} + \partial_\nu g_{\xi\lambda} - \partial_\xi g_{\nu\lambda}\right), \end{aligned} \tag{61}$$

then, after the substitution of (61) into Einsteins equations (56) left sides, and of relations (59) and (60) into these equations right sides, in addition to the electrodynamics equations (58) we get a closed system of equations relative to current J^μ, potential A^μ and metric tensor $g^{\mu\nu}$ components.

Formulation and proving of some theorems concerning an existence and a number of singular stationary solutions of this system, as well as elaboration of numerical methods of its solutions, is, undoubtedly, a difficult thing[8]. It might be one of the greatest challenges that mathematical physics has ever come across with.

The developed interpretation, which geometrically absorbs gravitation into currents and potentials electrodynamics, has its advantages and disadvantages, which we tried to summarize in the table.

Table 1

8 "It is a difficult thing, and no real mathematician would deal with it", – Einstein once said about the mathematical challenges of the unified field theory ([13], letter dated by April 9, 1947).

Strong points of taking gravitation into account in the form of geometric background of current and potential physics	**Weak points of taking gravitation into account in the form of geometric background of current and potential physics**
1. Gravitation is naturally and inevitably included into microphysics, particle physics, being no more just a "macrophysics makeweight" to microworld physics.	1. Gravitation constant G is built into the theory "by hand" as a fundamental empirical constant without any hope for its possible theoretical calculation in future.
2. The given equation system (56), (58), (59), (60), (61) as much as possible satisfies the ideal of theoretical physics, which Einstein once called "Einheitliche Feldtheorie von Gravitation und Electrizitat" (unified field theory), if we refuse persistent attempts of the author of this ideal to geometrize electrodynamics and to put up with Einstein's geometrization of gravitation, like with inevitable evil and currently common approach.	2. Geometrized Einstein's interpretation of gravitation without any hope for introducing gravitation into the theory in common physical language, language of currents and potentials (fields in flat Minkowski's space) is accepted without any analysis and critics; gravitation, turning into geometrical background, is forever removed beyond the frames of physics - the theory of dynamically conjugate currents and potentials interaction.

10. Estimation of non-dimensional gravitation constant k. Fundamental challenge for the classical theory

To estimate nondimensional gravitation constant k by formula (57) we must set ourselves the goal to find the value of unknown fundamental length constant r_0. If we accept the conservative estimation, $r_0 \approx 10^{-16}$, then $k \approx 0.5 \cdot 10^{-34}$. With such a small value of constant k, a considerable deviation from Minkowski's flat geometry appears only in the zone, where non-dimensional values of 4-current J components reach the value of the order 10^{17}. With previously determined current behavior close to singularity line $J \propto r^{-1}$, it means that characteristic length of the zone with geometrical curvature is no more than $10^{-17} r_0$ - i.e. Planckian length order.

If we accept a radical assumption that fundamental length r_0 coincides with Planckian length r_p:

$$r_0 = r_p = \left(G\hbar / c^3 \right)^{1/2},$$

then

$$k = 8\pi\alpha, \tag{62}$$

where α is the fine structure constant, $\alpha = e^2/\hbar c$.

With such choice, gravitational and electromagnetic interactions have the same intensity and are described by the same constant of interaction α - which is, undoubtedly, an inspiring fact. In accordance with (62) $k \approx 0.2$, the geometry inside the current zone is considerably different from the flat one with all $J = 0(1)$ - the singular line domain of influence is not vanishingly small as compared to full 3-volume lepton, covering all this volume and the blundering space outside it.

Anyway, both with radical and conservative estimation of the fundamental length r_0, these estimations lead us to a tough and unfavorable conclusion: *any correctly made classical theory must deal with distances of Planckian length order, maintaining its applicability right up to such short distances, and, therefore, must be quantum, but not classical* - or, at least, it must contain something which is conceptually close to the naive rules of quantization by Bohr type. This is a genuine apory ($\alpha\pi o\rho\iota\alpha$) - almost antique by tragic intensity. The only possible rational form of reaction to this problem nowadays, with absence of quantum version of theory, consists of ignoring the problem of space curvature, as well as the problem of infinite lepton mass, and of the attempt to extract maximal possible number of mathematical and conceptual effects of the classical theory developing in flat space.

11. Current restrictions: the way to avoid the appearance of singular current lines

The reasoning, provided above in p.6, that the singularity of solution (and the infinity of own mass of particles) is not a drawback of the *classical* theory, it transfers the responsibility for obtaining a finite value of mass to the object of the quantum theory. This reasoning can be criticized from the following positions. The procedure of quantization of the theory (whatever way we would use to carry it out) can turn the continuous quantity own mass into a discrete one, but hardly can it turn *the infinite* into *finite.* If we want to get the finite value of mass in the (nonexistent yet) quantum version of the theory of non-point particles, we have to provide the finiteness of mass in the classical version of the theory developed here, as well.

Elimination of singular current lines from the theory developed here is quite costly: we have to introduce into the theory one more fundamental constant j, which has the dimension of current density. We will add the following postulate to the theory: the negative pseudo-Euclidean square of current density is limited from below:

$$J^\nu J_\nu \geq -j^2. \tag{63}$$

The sign of equality in inequality (63) is reached on some inner tube-like boundaries of the area filled with currents. These tube-like inner boundaries are an integral part of pomerium. There is no current inside these tubular boundaries.

Thus introduced new fundamental constant j, due to its status in physics, would replace an electron charge which would thus turn from a fundamental constant into computable physical quantity expressed through j, c and r_0.

The appearance of fundamental current density in the continual theory of currents and potentials developed here is, in fact, expected. Currents are primary; particles with their masses and charges are the product of the theory, the eigenstates of current fields.
Formula (51), given above as the basis for muon mass calculation, makes a certain sense for the theory that contains limiting (63). However, application of this formula in such version of the theory does not require the limiting process $\varepsilon \longrightarrow 0$: the dimensions and the form of cavitated tubes have to be determined numerically; in the course of construction of the solution to a stationary problem, they obey the condition (63).
Current limiting (63) causes limitation of components of energy-momentum tensor, but does not remove the necessity to account space-time curvature described above: near the boundaries of "cavitated" tubes, the right sides of the Einstein equations (56) become the quantities of the order of unity[9].

12. Stationary problem in arbitrary reference system. The 4-velocity

The stationary problem is formulated in rest frame K_s. In this system conserving 4-momentum $P^{\mu}_{(s)}$ has the following view:

$$P^{\mu}_{(s)} = \{m, 0, 0, 0\}. \tag{64}$$

We shall view m as some symbol, ignoring the fact that lepton m own weight can be infinite within the frames of the theory, if we do not take into account condition (63).
In arbitrary Lorentz reference system K, 4-momentum is defined according to its vectorial features by the following relation:

$$P^{\mu} = \lambda^{\mu}_{\nu} P^{\nu}_{(s)} = m\lambda^{\mu}_{0}, \tag{65}$$

where λ^{μ}_{ν} is the matrix of Lorentz-rotation from K_s to system K. We shall denote

$$u^{\mu} = \lambda^{\mu}_{0}, \tag{66}$$

which gives:

$$P^{\mu} = mu^{\mu}. \tag{67}$$

The quantity u^{μ} can be called 4-velocity of the charge with respect to system K.
Visible Lorentz-noninvariance of definition (66) is connected with the distinguished nature of the rest frame. It is not difficult to make sure that 4-velocity is 4-vector. Really, let

[9] Introduction of the current limiting postulate (63) is undoubtedly, a gesture of despair rather than an action dictated by clear physical intuition. However, the history of physics has such "gestures". The introduction of the idea of discrecity in radiation / absorption processes by Max Planck in December 1900 was the gesture of despair dictated by aspiration to avoid ultra-violet catastrophe. The introduction of neutrino by Wolfgang Pauli in 1930 was the gesture of despair dictated by aspiration to non-violation of the law of energy conservation in the processes of β-decay. Fitzgerald's shortening once was also an act of despair.

us assume that K' is some reference system different from K_s and K. 4-momentum P^μ in system K' is defined by the expression (65):

$$P'^\mu = m\lambda_0'^\mu = mu'^\mu,$$

where $\lambda_\nu'^\mu$ is the matrix of Lorentz-rotation from K_s to system K'.
But matrix λ can be represented in the form of the product of Lorentz matrixes λ' and Λ, where Λ is the matrix of rotation from K' to system K:

$$\lambda_\nu^\mu = \Lambda_\xi^\mu \lambda_\nu'^\xi.$$

So,

$$u^\mu = \lambda_0^\mu = \Lambda_\nu^\mu \lambda_0'^\nu = \Lambda_\nu^\mu u'^\nu,$$

which means that u^μ is a 4-vector by its transformational properties.
The known identity $u^\mu u_\mu = 1$ follows from pseudo-orthogonality of Lorentz-rotation matrix:

$$u^\mu u_\mu = \mathrm{g}_{\mu\nu} u^\mu u^\nu = \mathrm{g}_{\mu\nu} \lambda_0^\mu \lambda_0^\nu = \mathrm{g}_{00} = 1.$$

We shall call an arbitrary point, stationary in the rest frame, a "charge center" - for example, the point, relative to which a dipole moment vanishes. The world line of center x_c^μ in the rest frame coincides with time axis . In arbitrary reference system K it is calculated by Lorentz-transform

$$x_c^\mu = \lambda_\nu^\mu x_{c(s)}^\nu = \lambda_0^\mu t_s = u^\mu t_s. \tag{68}$$

From (68) follows that

$$u^\mu = \frac{\mathrm{d}x_c^\mu}{\mathrm{d}t_c}. \tag{69}$$

Relations (64)(67) demonstrate that in relativistic mechanics of non-point charge, momentum is the main concept, connected only with the tensor of field energy, and 4-velocity is nominally determined by (66). As far as it is known, the situation in mechanics of point particles is the opposite [6]: formula (69) is viewed as determination of the fundamental quantity velocity (by indefinable idea of the world line of a point object), and relation (67) is viewed as definition of momentum through velocity with help of constant multiplier m, which physical interpretation is determined only by the principle of accordance with Newton mechanics. Formulation of the theory, developed here, does not require any obvious reference to Newton's ideas.

13. Charge polarization. Spin-velocity vector

The information about charge orientation is focused in pure spatial components of angular momentum tensor $M^{\mu\nu}$:

$$M^{\mu\nu} = \int \left((x^\mu - x_c^\mu) \left(T_{cur}^{\nu\lambda} + T_f^{\nu\lambda} \right) - (x^\nu - x_c^\nu) \left(T_{cur}^{\mu\lambda} + T_f^{\mu\lambda} \right) \right) \mathrm{d}\Sigma_\lambda, \tag{70}$$

where the integral is taken on arbitrary unclosed time-like hyper-surface Σ, which contains all infinitely remote points.
Space-like spin pseudo-vector a^μ [14] can be used instead of tensor $M^{\mu\nu}$:

$$a^\mu = -\frac{1}{2}\varepsilon^{\mu\nu\lambda\xi}M_{\nu\lambda}u_\xi; \qquad M_{\mu\lambda} = \varepsilon_{\mu\nu\lambda\xi}a^\lambda u^\xi. \tag{71}$$

In the rest frame the components of this vector ("Pauli-Lubansky vector") have the following view:

$$a^\mu_{(s)} = \{0, \mathbf{M}\}, \tag{72}$$

where $\mathbf{M}$ is intrinsic moment 3-vector.
It is appropriate to direct axis x_3 of system K_s along vector $\mathbf{M}$; then

$$a^\mu_{(s)} = M\delta^\mu_3, \tag{73}$$

where M is vector$\mathbf{M}$ modulus, δ^ν_μ is Kronecker symbol. The components of vector a^μ in arbitrary reference system are obtained with help of Lorentz-transformation (73):

$$a^\mu = \lambda^\mu_\nu a^\nu_{(s)} = M\lambda^\mu_3. \tag{74}$$

We shall denote

$$\lambda^\mu_3 = v^\mu, \tag{75}$$

then

$$a^\mu = Mv^\mu. \tag{76}$$

Formula (76) is the analog of the appropriate formula for momentum (67).
It is easy to make sure that v^μ is 4-vector. By analogy with vector u^μ it can be named a spin velocity vector. From (72) and (76) follows that $v^\mu v_\mu = -1$. The orthogonality relation $u^\mu v_\mu = 0$ is also obvious.
4-vector of magnetic moment μ^ν can be also considered in the capacity of integral characteristic of stationary solutions. We shall formulate it in the following way. In the rest frame:

$$\mu^\nu_{(s)} = \{0, \boldsymbol{\mu}\}, \tag{77}$$

where $\boldsymbol{\mu}$ is 3-vector of magnetic moment. In arbitrary system:

$$\mu^\nu = \lambda^\nu_\xi \mu^\xi_{(s)}. \tag{78}$$

If vectors μ and $\mathbf{M}$ are parallel in the rest frame, then it follows from (78) and (76) that $\mu^\nu = \mathrm{g}a^\nu$, where $\mathrm{g} = \mu/M$ is a gyromagnetic multiplier.

14. Charge in weak external field

We shall give consideration to approximate interpretation of evolutionary problem (one charge in the external, preassigned electromagnetic field). External field $F^{\mu\nu}_{(e)}$ will be considered weak in the following sense. Let us assume that in the neighborhood of each

infinite time-like hyper-surface Σ ("in each moment of time") we can introduce a local rest frame $\widetilde{K}$ depending on Σ (on time), such one in which

$$\widetilde{J}^{\mu} = J^{\mu}_{(s)}, \tag{79}$$

where $\widetilde{J}^{\mu}$ is the current in system $\widetilde{K}$, and $J^{\mu}_{(s)}$ is the stationary problem solution in the rest frame.

Condition (79) means that external field does not cause disturbance into the current distribution, but just shifts this distribution with the course of time.

In arbitrary fixed reference system K, current is estimated by the relation

$$J^{\mu} = \lambda^{\mu}_{\nu} J^{\mu}_{(s)}, \tag{80}$$

where λ is Lorentz-rotation matrix from $\widetilde{K}$ to K depending on the surface Σ ("on time"). Separating the self-field and the external field, we shall formulate the equation of conservation of energy-momentum (4) in the following form:

$$\partial_{\nu} T^{\mu\nu}_{cur} = F^{\mu\nu} J_{\nu} + F^{\mu\nu}_{(e)} J_{\nu}, \tag{81}$$

where, in this case, $F^{\mu\nu}$ is a self electromagnetic field tensor of stationary currents, represented in system K; $T^{\mu\nu}_{cur}$ is an energy-momentum tensor of stationary currents.

Equation (81), in contrast to (4), is approximate. It is fairly local in the neighborhood of each 3-surface Σ.

For further convenience we shall transform (81) to the form

$$\partial_{\nu} \left(T^{\mu\nu}_{cur} + T^{\mu\nu}_{f} \right) \cong F^{\mu\nu}_{(e)} J_{\nu}, \tag{82}$$

where $T^{\mu\nu}_{f}$ is an energy-momentum tensor of the self (stationary) electromagnetic field, recorded in system K.

Let us integrate (82) by infinitely small volume $\Delta\Omega$ flanking to the given hyper-surface Σ. 4-volume can be imagined as a part of 4-space, restricted by two infinitely close time-like hyper-surfaces Σ and Σ_1, closing up at infinity. These two hyper-surfaces must be so close that the change of matrix λ, when passing from Σ to Σ_1, could be neglected. The integration gives the relation

$$\int_{\Delta\Omega} \partial_{\nu} \left(T^{\mu\nu}_{cur} + T^{\mu\nu}_{f} \right) \mathrm{d}\Omega = \int_{\Delta\Omega} F^{\mu\nu}_{(e)} J_{\nu} \mathrm{d}\Omega. \tag{83}$$

We shall transform the left side of (83) by Gauss theorem, taking into account the absence of energy flux at infinity in the stationary problem:

$$\int_{\Delta\Omega} \partial_{\nu} \left(T^{\mu\nu}_{cur} + T^{\mu\nu}_{f} \right) \mathrm{d}\Omega = \Delta P^{\mu}, \tag{84}$$

where $P^{\mu} = \int_{\Sigma} \left(T^{\mu\nu}_{cur} + T^{\mu\nu}_{f} \right) \mathrm{d}\Sigma$ is the total 4-momentum of charge on hyper-surface Σ, and ΔP^{μ} is increase of momentum at passing from Σ to Σ_1: $\Delta P^{\mu} = P^{\mu}(\Sigma_1) - P^{\mu}(\Sigma)$.

In the right side of (83) the external field can be taken outside integral symbol, if it has

little change within the frames of self-volume V_J, occupied by currents (otherwise, the usage of notion "external field" is unjustified). Finally, (83) takes on the form:

$$\Delta P^\mu \cong F^{\mu\nu}_{(e)} \int_{\Delta\Omega} J_\nu \mathrm{d}\Omega. \tag{85}$$

Taking into account the neighborhood of Σ and Σ_1, and Lorentz- invariance of 4-volume, 4-integral in the right side of (85) can be transformed into 3-integral by self-volume V_J:

$$\int_{\Delta\Omega} J_\nu \mathrm{d}\Omega \cong \Delta t_s \int_{V_J} J_\nu \mathrm{d}V_{(s)}, \tag{86}$$

where Δt_s is invariant proper time interval between Σ and Σ_1.
Using (80), we shall write integral (86) in the following way:

$$\int_{V_J} J_\mu \mathrm{d}V_{(s)} = \mathrm{g}_{\mu\nu}\lambda^\nu_\xi \int_{V_J} J^\xi_{(s)} \mathrm{d}V_{(s)}. \tag{87}$$

But the integral in (87) has a simple expression:

$$\int_{V_J} J^\xi_{(s)} \mathrm{d}V_{(s)} = e\delta^\xi_0.$$

Actually, integral (87) from time component of the current is, by definition, a total electric charge of particle e, but integrals from spatial components come to nothing on the account of stationary problem boundary conditions and the continuity equation.
So, (85) can be represented in the form:

$$\Delta P^\mu \cong e\mathrm{g}_{\nu\xi}\lambda^\xi_0 F^{\mu\nu}_{(e)}\Delta t_s, \tag{88}$$

and, as by (66): $\lambda^\xi_0 = u^\xi$, where u^ξ is 4-velocity of the charge with respect to system K, classical equation of point charge motion in the given electromagnetic field follows from (88) with help of limiting process $\Delta t_s \longrightarrow 0$:

$$\frac{\mathrm{d}P^\mu}{\mathrm{d}t_s} = eF^{\mu\nu}_{(e)}u_\nu. \tag{89}$$

15. Spin in weak external field

To study the changes of polarization in external field, we shall introduce the third rank tensor $M^{\mu\nu\lambda}$, anti-symmetric by the first two indexes:

$$M^{\mu\nu\lambda} = (x^\mu - x^\mu_c)\left(T^{\nu\lambda}_{cur} + T^{\nu\lambda}_f\right) - (x^\nu - x^\nu_c)\left(T^{\mu\lambda}_{cur} + T^{\mu\lambda}_f\right), \tag{90}$$

where $T^{\mu\lambda}_{cur}$ and $T^{\mu\lambda}_f$ are energy-momentum tensors of stationary current and stationary electromagnetic self field, represented in arbitrary Lorentz reference system.
In the approximation equal to (88), the appropriate equation for $M^{\mu\nu\lambda}$ can be formed:

$$\partial_\lambda M^{\mu\nu\lambda} \cong F^{\nu\lambda}_{(e)}\Delta x^\mu J_\lambda - F^{\mu\lambda}_{(e)}\Delta x^\nu J_\lambda, \tag{91}$$

where $\Delta x^\mu = x^\mu - x^\mu_{(c)}$. By integrating (91) by 4-zone of $\Delta\Omega$, like in the previous item, at formulation of equation of motion, we find out that

$$\frac{\mathrm{d}M^{\mu\nu}}{\mathrm{d}t_s} = F^{\nu\lambda}_{(e)}\mu^\mu_\lambda - F^{\mu\lambda}_{(e)}\mu^\nu_\lambda, \tag{92}$$

where $\mu^{\nu\lambda} = \int_{V_J} \Delta x^\nu J^\lambda \mathrm{d}V_{(s)}$. Using the connection (71) between moment tensor $M^{\mu\nu}$ and Pauli vector a^ν, as well as equation of motion (89), after a number of algebraic transformations, we find by (92):

$$\frac{\mathrm{d}a^\nu}{\mathrm{d}t_s} = 2\mu_\lambda F^{\nu\lambda}_{(e)} - 2u^\nu u_\lambda F^{\lambda\xi}_{(e)}\left(\mu_\xi - \frac{e}{2m}a_\xi\right), \tag{93}$$

where μ^λ is 4-vector of magnetic moment, introduced by relations (77), (78). As $\mu^\lambda = \mathrm{g}a^\lambda$, (93) can be represented in the form which is analogous to the famous Bargmann-Michel-Telegdi [14] equation

$$\frac{\mathrm{d}a^\mu}{\mathrm{d}t_s} = 2\mathrm{g}F^{\mu\nu}_{(e)}a_\nu - 2\mathrm{g}'u^\mu u^\nu F_{\nu\lambda}a^\lambda, \tag{94}$$

where $\mathrm{g}' = \mathrm{g} - \dfrac{e}{2m}$.

Quantity g', by analogy with the appropriate quantity in modern formulation of quantum electrodynamics, can be called "anormal" gyro-magnetic ratio. However, within the frames of the theory being developed, this quantity does not contain anything abnormal. Appropriate "anormal" magnetic moment μ' , as follows from (76) and (78), is defined by the inherent moment momentum of the particle: $\mu' = \mathrm{g}'M = \mu - \dfrac{eM}{2m}$. Combination $\dfrac{eM}{2m}$, which has classical origin, is the analogue of absolutely quantum quantity - Dirac magnetic moment $\dfrac{e\hbar}{2m}$ ($\hbar$ is Planck constant).

"Deducibility" of motion equations (88) and (93) is, in some way, trivial – it demonstrates only the possibility of approximate preserving the concept of discrete particle in the field theory and does not rest on any specific kind of energy-momentum tensor (5). However, inherent characteristics of m, M and μ, included into definition of momentum, vector a^μ and gyromagnetic relation, depend on the kind of tensor $T^{\mu\lambda}_{cur}$.

16. Problems without current boundaries: heavy photon

Within the frames of the formulated theory we can study such problems, in which the boundary of current zone Ω_J, determined by the condition of isotropy of current 4-vector (14), is absent. The trivial example of such problem is: free from currents space-time, filled with free electromagnetic fields ($J^\mu \equiv 0, A^\mu \neq 0$) that obey Maxwell's homogeneous equations. Particular solution in the form of plane monochromatic wave is a transverse wave with isotropic wave 4-vector $k^\mu = \{\omega, \mathbf{k}\}$ [6]

$$k^\mu k_\mu = 0, \tag{95}$$

or, in 3- notations:

$$\omega^2 = \mathbf{k}^2. \tag{96}$$

Dispersive relation (95), (96) describes the wave without dispersion, spreading with unit velocity (both phase velocity and group velocity are equal to unit). With usual quantum interpretation, a massless particle – photon – corresponds to dispersive relation (95). However, the theory under consideration also contains a new nontrivial wave, in which the field of 4-current vector exists all over 4-space. Equations (31), describing the problem in Lorentz gauge, can be represented in the following form:

$$\begin{array}{ll} \Box J^\mu = 4\pi J^\mu; & \text{(a)} \\ \partial_\mu J^\mu = 0; & \text{(b)} \\ J^\mu J_\mu \leq 0; & \text{(c)} \\ A^\mu = -J^\mu. & \text{(d)} \end{array} \tag{97}$$

Seeking for solution (97) in the form of plane monochromatic wave:

$$J^\mu = Qn^\mu \cos\phi,$$

where $\phi = k^\mu x_\mu$ – Lorentz-invariant phase of wave, Q is an arbitrary scalar amplitude multiplier, n^μ is 4-vector, defining current field orientation. From the condition of current space-likeliness (97) follows that n^μ is a unitary space-like vector:

$$n^\mu n_\mu = -1. \tag{98}$$

The condition of wave transversality follows from the law of current conservation (97 b):

$$n^\mu k_\mu = 0, \tag{99}$$

and dispersive relation for the current electromagnetic wave under consideration follows from wave equation(97 a):

$$k^\mu k_\mu = 4\pi, \tag{100}$$

or, in 3-notations:

$$\omega^2 - \mathbf{k}^2 = 4\pi. \tag{101}$$

Dispersive relation (100), (101) describes a wave with phase velocity (k is a module of wave 3-vector $\mathbf{k}$):

$$\nu_{ph} = \frac{\omega}{k} = \frac{\sqrt{4\pi + k^2}}{k} > 1$$

and with group velocity:

$$\nu_{\mathrm{g}} = \frac{\mathrm{d}\omega}{\mathrm{d}k} = \frac{k}{\sqrt{4\pi + k^2}} < 1.$$

With usual quantum interpretation, dispersive relation (101) describes the particle with non-vanishing rest mass m_h:

$$m_h = \sqrt{4\pi},$$

or, in dimensional form:

$$m_h = \sqrt{4\pi}\frac{\hbar}{r_0 c}, \tag{102}$$

This particle can be called a "heavy photon" (or "massive photon"). With conservative evaluation $r_0 \approx 10^{-16}$ cm, according to (102), heavy photon mass is no less than 700 GeV. This mass exceeds the masses of the heaviest known particles- intermediate bosons and t-quark-, but lies in the area accessible for modern accelerators.

If we accept the radical evaluation of fundamental length r_0 ($r_0 = r_p$), heavy photon mass goes to laboratory unattainable Planck's area:

$$m_h = \sqrt{4\pi\frac{\hbar c}{G}} \approx 4 \cdot 10^{19}\mathrm{GeV}.$$

Remaining within the frames of the classical wave interpretation of the obtained solution, we should emphasize that it describes a diffusing wave, a wave with dispersion.

It is convenient to describe the structure of "heavy photon" wave in its own reference system, where a wave 3-vector $\mathbf{k}$ comes to nothing (the wave does not spread), and $\omega = \omega_s = \sqrt{4\pi}$ (in accordance with dispersive relation (101)). In this reference system all physical quantities do not depend on coordinates, but only depend on time: the wave "flashes" and "dies out" simultaneously in the whole three-dimensional space. Our Lorentz' intuition must not protest violently against such situation: creation of flat monochromatic wave, occupying the whole 3-space, requires unlimited amount of energy; we can consider such wave only as a single Fourier component of some wave packet which has a non-zero extent in the space of wave vectors $\mathbf{k}$; such packet has finite energy, it is localized in space, and besides this, being dispersive, it gradually diffuses.

In own reference system, according to (99):

$$\begin{aligned} &n_s^\mu = \{0; \mathbf{n}_s\}; \\ &\mathbf{n}_s^2 = 1. \end{aligned} \tag{103}$$

In own reference system the time component of 4-current J_s^0 and scalar potential φ_s amounts to zero, magnetic field $\mathbf{H}_s$ is absent, and current $\mathbf{J}_s$, 3-vector potential $\mathbf{A}_s$ and electric field $\mathbf{E}_s$ are calculated by simple relations:

$$\begin{aligned} &\mathbf{J}_s = Q\mathbf{n}_s \cos\phi, \\ &\mathbf{A}_s = -Q\mathbf{n}_s \cos\phi, \\ &\mathbf{E}_s = -\omega_s Q\mathbf{n}_s \sin\phi, \\ &\phi = \omega_s t_s. \end{aligned} \tag{104}$$

The energy density of electromagnetic field in this system is

$$w_{f,s} = \frac{\mathbf{E}_s^2 + \mathbf{H}_s^2}{8\pi} = \frac{1}{2}Q^2 \sin^2\phi. \tag{105}$$

And energy density of current field is:

$$w_{cur,s} = T^{00}_{cur,s} = \frac{1}{2}Q^2 \cos^2\phi. \tag{106}$$

In accordance with expressions (105) and (106), current energy constantly turns into electromagnetic field energy, but total energy density does not change here:

$$w_s = w_{f,s} + w_{cur,s} = \frac{1}{2}Q^2 = \text{const}.$$

Description of this wave in arbitrary reference system, relative to which heavy photon moves with three-dimensional velocity $\mathbf{v}$:

$$\mathbf{v} = \text{th}\psi\{1, 0, 0\},$$

ψ, an arbitrary real-valued parameter, can be obtained with help of Lorentz- transformation of solution (104), obtained in intrinsic system [6]. Making this transformation, we find that

$$\begin{aligned}
&J^0 = Qn_{1,s}\text{sh}\psi\cos\phi,\\
&J^1 = Qn_{1,s}\text{ch}\psi\cos\phi,\\
&J^2 = Qn_{2,s}\cos\phi,\\
&J^3 = Qn_{3,s}\cos\phi,\\
&\phi = \omega t - \mathbf{kx},\\
&\omega = \omega_s\text{ch}\psi,\\
&k_x = \omega_s\text{sh}\psi,\\
&k_y = 0, \quad k_z = 0.
\end{aligned} \tag{107}$$

By transforming 3-vectors of electrical field $\mathbf{E}$ and magnetic field $\mathbf{H}$ from rest system to arbitrary system, we find

$$\begin{aligned}
&E_{\|} = -\omega_s Q n_{1,s}\sin\phi,\\
&H_{\|} = 0,\\
&\mathbf{E}_{\perp} = Q\omega_s\mathbf{n}_{\perp}\sin\phi,\\
&\mathbf{H}_{\perp} = Q\mathbf{k}\times\mathbf{n}_s\sin\phi.
\end{aligned} \tag{108}$$

In expressions (108) indexes II and $\perp$ refer to longitudinal part of the field (in the line of velocity $\mathbf{v}$ and wave vector $\mathbf{k}$) and crosscut part of the field, respectively. From formulas (108) it is obvious that in "heavy photon", vectors $\mathbf{E}$ and $\mathbf{H}$ are orthogonal to each other, like in an ordinary electromagnetic wave.

Calculation of wave energy density w, as a sum of current energy and electromagnetic field energy, shows that energy can be represented as a sum of a stationary component and a variable component:

$$w = w_{\text{const}} + w_{\text{var}},$$

where

$$\begin{aligned}
w_{\text{const}} &= \frac{1}{2}Q^2\text{ch}^2\psi,\\
w_{\text{var}} &= \frac{1}{2}Q^2\text{sh}^2\psi\cdot\cos 2\vartheta\cdot\cos 2\phi.
\end{aligned} \tag{109}$$

In this expression, angle ϑ is an angle between vectors $\mathbf{n}$ and $\mathbf{k}$. Calculating vector of energy flux density $\mathbf{S}$ in the wave of "heavy photon" as the sum of Pointing vector $\mathbf{S}_f$

and vector of energy flux density of current field $\mathbf{S}_{cur}$:

$$\mathbf{S}_{cur} = J^0\mathbf{J},$$

we find that there are three energy flux components: a) constant crosscut energy flux

$$\mathbf{S}_{\perp} = \frac{1}{2}Q^2 \mathrm{sh}\psi \cos\vartheta \cdot \mathbf{n}_{\perp},$$

b) constant longitudinal energy flux

$$\mathbf{S}_{\parallel} = \frac{1}{2}Q^2 \mathrm{sh}\psi \cos\vartheta \cdot \mathbf{i},$$

($\mathbf{i}$ – axes x unit vector).
By using the expression for w_{const} from (109), we can represent this vector by the following form

$$\mathbf{S}_{\parallel,\mathrm{const}} = w_{\mathrm{const}} \cdot \mathbf{v},$$

which demonstrates an obvious energy transfer towards "heavy photon" with the velocity of its motion
c) variable longitudinal energy flux

$$\mathbf{S}_{\parallel,\mathrm{var}} = \frac{1}{2}Q^2 \mathrm{sh}\psi \mathrm{ch}\psi \cos\vartheta \cos 2\phi \cdot \mathbf{i}.$$

It is not difficult to make sure that a variable part of energy density and a variable part of energy flux satisfy the energy conservation law, as it was expected

$$\partial_t w + \mathrm{div}\mathbf{S} = 0.$$

In conclusion, let us take notice of the fact that heavy photon makes us return to the classical problem, which was successfully solved by Max Planck in 1900, – the problem of electromagnetic field thermodynamics. Planck's distribution stops being correct in the high-frequency zone, $\omega > \omega_s$, in the frequency zone, in which heavy photon exists side by side with an ordinary photon.

17. Problems without current boundaries: Maxwell neutrino

The developed theory contains, as a special case, such situation when 4-current is isotropic not on the isolated 2-surface, which is a moving boundary of the current tube, but in some 3-zone:

$$J^\nu J_\nu = 0, \qquad J^\nu \neq 0. \tag{110}$$

We shall call the current, satisfying condition (110) in some 3D-region, a "neutrino current".
We shall immediately notice that the problem of neutrino current cannot be contemplated correctly in Maxwell's sector of physical theory. Neutrino current, satisfying relation

(110), automatically becomes not only Maxwell's current, but it also transforms into one of the components of Yang-Mills triplet of weak currents, and it generates Yang-Mills triplet of potentials. Neutrino current "does not fit" into Maxwell's sector of physics, mixing Maxwell's singlet sector with Yang-Mills triplet one.

However, we have the right to turn back to the epoch preceding emergence of C. N. Yang and R.L. Mills article [15]. We can regard the model of Maxwell's neutrino in the way "as if the weak interaction did not exist", or, as Hugo Grotius once said: "etsi Deus non daretur" (as if God did not exist [16]). If this work had been written in the 30th-40th of the XXth century, when conceptual and mathematical instrumentality for it had already actually existed in P. Diracs articles [7], [10], the neutrino state would have been regarded exactly in this way.

Real neutrino in some way would have been close to Maxwell's neutrino field in the world with extreme critical values of Weinberg angle ϑ_w, with $\vartheta_w \longrightarrow \pi/2$ and $\vartheta_w \longrightarrow 0$. In the first case, the contribution into Lagrangian from interaction of Maxwell's singlet current with Maxwell's singlet potential would have significantly exceeded the contribution from interaction of Yang-Mills currents triplet with Yang-Mills potentials triplet, and the world would be "more Maxwell's than Yang-Mills"; in the second case, the interaction of potentials inside Yang-Mills triplet would have been weak, and the triplet would have disintegrated into three almost independent, almost Maxwell's singlet.

Weinberg angle, as well as the fine structure constant and gravitation constant, are empirical parameters in modern physics: we do not know why God gave them these very values but not some different ones. We are free to discuss worlds systems with different values of these parameters.

Neutrino condition probably emerges as a result of the loss of lepton stationary condition stability, which is accompanied with two-dimensional boundary of massive lepton "swelling" and its transformation into three-dimensional zone, filled with neutrino current. Such zone is knowingly non-stationary, its boundary moves. Not having any mathematical description of this process of instability increase, we shall consider the model problem, assuming that neutrino current region has no boundaries either in space, or in time.

For neutrino condition, there is no current contribution into Lagrangian density ($J^\nu J_\nu = 0$), and the base Lagrangian of neutrino problem has the following form:

$$L = -J^\mu A_\mu - \frac{1}{16\pi} F^{\mu\nu} F_{\mu\nu}. \tag{111}$$

The action functional must be minimized with regard for condition of neutrino (110) and current conservation (3). Account of these conditions with help of Lagrange multipliers gives addition to Lagrange (111):

$$L_{ad} = -\frac{\lambda}{2} J^\mu J_\mu - J_\mu \partial^\mu \chi, \tag{112}$$

where λ and χ are unknown functions of coordinates and time (Lagrange multipliers). As it was noted above, without the loss of generality, it is possible to assume that $\chi = 0$

(Lorentz gauge).
Effective Lagrangian is the sum of expressions (111) and (112). Minimization of action with this Lagrangian gives the following field equations, justified for the whole 4-space:

$$\begin{aligned} &\lambda J^\mu + A^\mu = 0, \\ &A^\mu + \partial^\mu l = -4\pi J^\mu, \\ &J^\mu J_\mu = 0, \\ &\partial_\mu J^\mu = 0. \end{aligned} \tag{113}$$

Like in the previous case, in equations (113) l is a 4- divergence of potential A^ν.
Let us study the particular solution of system (113) with the constant value of parameter λ. If $\lambda = \text{const}$, then $l = 0$ and the equation system can be represented in the following view:

$$\begin{aligned} &\Box A^\mu = -4\pi a A^\mu, && \text{(a)} \\ &\partial_\mu A^\mu = 0, && \text{(b)} \\ &A_\mu A^\mu = 0, && \text{(c)} \\ &J^\mu = a A^\mu, && \text{(d)} \end{aligned} \tag{114}$$

$(a = -1/\lambda)$.
Searching for solution of this problem in the phase of a plane monochromatic wave with plane polarization (condition of isotropy, imposed in (114) on potential vector A^μ, excepts the existence of waves with elliptical polarization), let us assume that

$$A^\mu = Q n^\mu \cos\phi, \tag{115}$$

where ϕ is Lorentz-invariant phase of wave

$$\phi = k^\mu x_\mu,$$

Q is an arbitrary amplitude multiplier, n^μ is a constant isotropic vector, defining potential A^μ orientation:

$$n^\mu n_\mu = 0, \tag{116}$$

which can be standardized, without loss of generality, with the condition

$$n^0 = 1. \tag{117}$$

Considering (116) and (117), spatial part of 4-vector n^μ is some unit 3-vector $\mathbf{n}$. From condition (114b) follows that orthogonality of wave 4-vector k^μ and isotropic unit vector n^μ:

$$k^\mu n_\mu = 0. \tag{118}$$

In three-dimensional notation, (118), with the account of (117), has the following view

$$\omega - \mathbf{k}\cdot\mathbf{n} = 0. \tag{119}$$

From relation (119) follows that neutrino wave frequency does not exceed the three-dimensional wave vector magnitude:

$$\omega \leq\mid \mathbf{k} \mid,$$

consequently,

$$\omega^2 - k^2 \leq 0,$$

or, in 4-notation:

$$k^\mu k_\mu \leq 0. \tag{120}$$

Condition (120) means that wave vector of neutrino wave is space-like. Substitution of the solution form (115) into the wave equation (114 a) gives dispersive relation for neutrino wave:

$$\begin{aligned} k^\nu k_\nu &= -4\pi a, \\ \omega^2 - \mathbf{k}^2 &= -4\pi a. \end{aligned} \tag{121}$$

Condition (120) means that $a > 0$. With formal quantum interpretation of dispersive relation (121), it should be concluded that it describes a tachyon - a particle with negative mass square m_N:

$$m_N^2 = -4\pi a.$$

In accordance with dispersive relation (121), neutrino wave phase velocity $v_{ph} = \dfrac{\omega}{k}$ is less than the velocity of light, but group velocity $v_{\mathrm{g}} = \dfrac{\mathrm{d}\omega}{\mathrm{d}k}$, usually interpreted as the velocity of energy transfer in wave, exceeds the velocity of light. However, with help of direct calculations, we shall discover that the velocity of energy transfer in neutrino wave does not exceed the velocity of light.

It is appropriate to study Maxwell's neutrino wave in intrinsic frame of reference in which wave frequency ω comes to nothing, and wave vector $\mathbf{k}$ is directed along the axis x. Its modulus k_s in intrinsic frame of reference will be defined from (121):

$$k_s = \sqrt{4\pi a}.$$

According to dispersive relation (121), k_s is the least possible value of wave number k.

In the intrinsic frame of reference, Maxwell's neutrino wave corresponds to motionless ripple, which crests are orthogonal to axis x. This ripple is the answer to the question, which a sixteen year-old Albert Einstein asked himself at the cantonal school of a Swiss city of Aarau: "What will happen if one runs after the light wave with the velocity of light?" [17]. It is impossible to catch up the light wave, but it is possible to sit into the rest frame of Maxwells neutrino (if to divert one's attention away from the problem of actual nonexistence of such object) and to see the standing-still, motionless ripple.

In the intrinsic wave system:

$$\begin{aligned}
&\phi = -k_s \cdot x,\\
&\mathbf{n} = \{0; n_y; n_z\}, \quad n_y^2 + n_z^2 = 1,\\
&\varphi = A^0 = Q\cos\phi,\\
&\mathbf{A} = Q\cos\phi \cdot \mathbf{n},\\
&\rho = J^0 = aQ\cos\phi,\\
&\mathbf{J} = aQ\mathbf{n}\cos\phi.
\end{aligned} \tag{122}$$

The electric field on neutrino wave in intrinsic system is longitudinal:

$$\begin{aligned}
&E_{\|} = -k_s Q\sin\phi,\\
&E_{\perp} = 0.
\end{aligned} \tag{123}$$

Magnetic field of neutrino wave is transverse:

$$\mathbf{H} = Q\sin\phi\;\mathbf{k}\times\mathbf{n}, \tag{124}$$

where $\mathbf{k} = k_s\{1;0;0\}$.
If we calculate the density of energy of neutrino wave w (as the sum of the current and electromagnetic contribution) and the density vector of energy flux in neutrino wave $\mathbf{S}$ (as the sum of the current and electromagnetic contribution) by formulas (122), (123) and (124), it is not difficult to ascertain that the following relation is performed:

$$\mathbf{S} = w\cdot\mathbf{v}; \quad \mathbf{v} = \mathbf{n}; \quad \mathbf{v}^2 = 1. \tag{125}$$

From relation (125) follows that the energy is transferred in neutrino wave with the velocity of light in the direction which is orthogonal to wave vector $\mathbf{k}$.
The expression looks especially simple for w and $\mathbf{S}$ in case, if voluntary production parameter a is assigned the value equal to one. With such choice of parameter a, 4-current J^ν in neutrino wave coincides with 4-potential A^ν, and energy parameters of the wave w and $\mathbf{S}$ do not depend on coordinate x:

$$w = Q^2; \quad \mathbf{S} = Q^2\mathbf{n}.$$

We shall convert the received solution to the arbitrary reference frame, with respect to which the intrinsic system of neutrino wave moves with the velocity of $v = \mathrm{th}\psi$ along wave vector $\mathbf{k}$.
In this frame of reference the neutrino wave parameters look the following way (with $a = 1$)
• current and potential
$\rho = \varphi = Q\mathrm{ch}\psi\cos\phi$;
$A_{\|} = J_{\|} = Q\mathrm{sh}\psi\cos\phi$;
$\mathbf{A}_{\perp} = \mathbf{J}_{\perp} = Q\mathbf{n}\cos\phi$;
• phase

$\phi = \omega t - kx;$
$\omega = k_s \mathrm{sh}\psi;$
$k = k_s \mathrm{ch}\psi;$
• energy density
$w = Q^2 \mathrm{ch}^2\psi = \mathrm{const};$
• energy flux vector
$\mathbf{S} = w \cdot \mathbf{v}; \quad \mathbf{v}^2 = 1;$
• energy transport velocity
$v_\parallel = \mathrm{th}\psi = u;$
$v_\perp = \dfrac{1}{\mathrm{ch}\psi} = \sqrt{1-u^2}.$
The latter relations demonstrate that the energy transfer velocity in the direction along the wave, as expected, coincides with the neutrino field motion as a single object, and the modulus of vector sum of longitudinal and transverse energy transfer velocity is equal to the velocity of light.
Some of the readers could say that too much attention is paid here to perhaps nonexistent object – "Maxwell's neutrino". However, the discussion of this object, as well as the discussion of "heavy photon" and the problems of stationary solutions singularity, is called to demonstrate the possibilities concealed in the formulated theory; the theory that could have been made as far back as in the $30^{\mathrm{th}} - 40^{\mathrm{th}}$ of the XXth century, if there was not the mental forbidding, formulated by A. Einstein [8] and shared by the majority of physicians: the empirically inaccessible details of electron internal structure should not be discussed; it is better to regard fundamental particles as point-like singularities.
On the whole, summarizing the discussion of the problems without current boundaries, we can see that within the frames of the formulated theory, the wave problems in Maxwell's sector of physics have a trifoliate dispersive relation:
1) $\omega^2 - k^2 = 0$ (photon).
2) $\omega^2 - k^2 = 4\pi$ ("heavy photon" probably, existing particle with the rest mass from 700 GeV (with $r_0 \approx 10^{-16}$ cm) to the range of Planck masses $\sim 10^{19}$ GeV).
3) $\omega^2 - k^2 = -4\pi$ (with $a = 1$) ("Maxwell's neutrino").

18. Concluding remarks

Quite many physicist-theorists, as a matter of principle, following Einstein [8], refuse to study any non-point (and non-string) classical models of elementary particles. Others, like Dirac, believe that "the troubles of the present quantum electrodynamics should be ascribed primarily $< \cdots >$ to our working from a wrong classical theory" [18]. The article [18] was written already after the works of Lamb, Schwinger, Feynman, after the creation of renormalization procedure, which makes it possible to remove divergences, and Dirac's evaluation of quantum relativistic theory as "un ugly and incomplete one" [18], formulated as far back as in 1951, is still true after six decades.
The classical electrodynamics of non-point particles, formulated in this article, without

full healing of the challenges of point particle *classical* electrodynamics (the full "healing" in classical theory is probably, in general, impossible), gives some hope for possible calculation of masses ratio of heavy leptons; predicts the existence of heavy photon; gives signals of the necessity of account of gravitation at classical models of elementary particles construction; demonstrates the possibility of Maxwell neutrinos existence.
But, more essentially from the conceptual point of view, this theory introduces dynamically conjugate pair of current and potential as a non-interpreted dual fundamental principle of matter; the particles become a product at the theory yield.
The attempt to pass this concept of dynamically conjugated pair of currents and potentials to Yang-Mills triplet of weak interactions and Yang-Mills octuplet of strong interactions seems to be reasonable. Currents and potentials, like Yin and Yan in Chinese philosophy, can make the basis of the whole construction of the *classical* version of the Standard Model. Gravitation is automatically included into the theory because of the singular lines presence in current fields (or because of large finite values of the components of energy-momentum tensor in the vicinity of "cavitated" tubes); in this theory Higgs fields are left no room for.

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